Homework 2

1. Section 19

19.1 Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.
(c) $f(x) = x^3$ on $(0, 1)$
(d) $f(x) = x^3$ on $\mathbb{R}$
(e) $f(x) = \frac{1}{x^3}$ on $(0, 1]$
(f) $f(x) = \sin \frac{1}{x^2}$ on $(0, 1]$.

19.2 Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the $\varepsilon - \delta$ property in Definition 19.1
(b) $f(x) = x^2$ on $[0, 3]$,
(c) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$.

19.4 (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. *Hint*: Assume not. Use Theorems 11.5 and 19.4.
(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

19.6 (a) Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show f is uniformly continuous on $(0, 1]$. (*Hint*: f is continuous on $[0, 1]$)
(b) Show f is uniformly continuous on $[1, \infty)$.

19.7 (a) Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some k, then f is uniformly continuous on $[0, \infty)$.
(b) Use (a) and Exercise 19.6(b) to prove $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

19.10 Let $g(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$.
(a) Observe g is continuous on $\mathbb{R}$; see Exercises 17.3(f) and 17.9(c).
(b) Why is g uniformly continuous on any bounded subset of $\mathbb{R}$?

2. Section 20

20.1/5 Let $f(x) = \frac{x}{|x|}$. Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0+} f(x)$, $\lim_{x\to 0-} f(x)$, $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to 0} f(x)$ when they exist. Also indicate when they do not exist. Prove the limit assertions.

20.4/8 Repeat Exercise [20.1] for $f(x) = x \sin \frac{1}{x}$.

20.16 Suppose the limits $L_1 = \lim_{x\to a+} f_1(x)$ and $L_2 = \lim_{x\to a+} f_2(x)$ exist.
(a) Show if $f_1(x) \leq f_2(x)$ for all x in some interval (a, b), then $L_1 \leq L_2$.
(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all x in some interval (a, b). Can you conclude $L_1 < L_2$?

20.17 Show that if $\lim_{x\to a+} f_1(x) = \lim_{x\to a+} f_3(x) = L$ and if $f_1(x) \leq f_2(x) \leq f_3(x)$ for all x in some interval (a, b), then $\lim_{x\to a+} f_2(x) = L$. This is called the squeeze lemma. *Warning*: This is not immediate from Exercise 20.16(a), because we are not assuming $lim_{x\to a+} f_2(x)$ exists; this must be proved.