Homework 4

1. Section 29

29.5 Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \le (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function.

29.7 (a) Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b.
(b) Suppose f is three times differentiable on an open interval I and $f'''(x) = 0$ on I. What form does f have? Prove your claim.

29.10 Let $f(x) = x^2 \sin(\frac{1}{x}) + \frac{x}{2}$ for $x \neq 0$ and $f(0) = 0$.
(a) Show $f'(0) > 0$; see Exercise 28.4.
(b) Show f is not increasing on any open interval containing 0.
(c) Compare this example with Corollary 29.7(i).

29.12 (a) Show $x < \tan x$ for all $x \in (0, \frac{\pi}{2})$
(b) Show $\frac{x}{\sin x}$ is a strictly increasing function on $(0, \frac{\pi}{2})$.
(c) Show $x \le \frac{\pi}{2} \sin x$ for $x \in [0, \frac{\pi}{2}]$.

29.14 Suppose f is differentiable on $\mathbb{R}$, $1 \le f'(x) \le 2$ for $x \in \mathbb{R}$, and $f(0) = 0$. Prove $x \le f(x) \le 2x$ for all $x \ge 0$.

2. Section 30

30.2 Find the following limits if they exist.
(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$
(c) $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x})$
(d) $\lim_{x\to 0}(\cos x)^{1/x^2}$

3. Section 23

23.1 For each of the following power series, find the radius of convergence and determine the exact interval of convergence.
(a) $\sum n^2 x^n$
(d) $\sum (\frac{n^3}{3^n}) x^n$
(e) $\sum (\frac{2^n}{n!}) x^n$
(h) $\sum (\frac{(-1)^n}{n^2 \cdot 4^n}) x^n$

23.4 For $n = 0, 1, 2, 3, \ldots$, let $a_n = [\frac{4+2(-1)^n}{5}]^n$.
(a) Find $\limsup (a_n)^{1/n}$, $\liminf (a_n)^{1/n}$, $\limsup |\frac{a_{n+1}}{a_n}|$ and $\liminf |\frac{a_{n+1}}{a_n}|$.
(b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Explain briefly.
(c) Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.