Homework 5

1. Section 23

23.6 (a) Suppose $\sum a_n x^n$ has finite radius of convergence R and $a_n \geq 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.
(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

23.7 For each $n \in \mathbb{N}$, let $f_n(x) = (\cos x)^n$. Each f_n is a continuous function. Nevertheless, show
(a) $\lim f_n(x) = 0$ unless x is a multiple of π,
(b) $\lim f_n(x) = 1$ if x is an even multiple of π,
(c) $\lim f_n(x)$ does not exist if x is an odd multiple of π.

23.8 For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n}\sin(nx)$. Each f_n is a differentiable function. Show
(a) $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$,
(b) But $\lim f_n'(x)$ need not exist [at $x = \pi$ for instance].

23.9 Let $f_n(x) = nx^n$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Show
(a) $\lim f_n(x) = 0$ for $x \in [0, 1)$. *Hint*: Use Exercise 9.12.
(b) However, $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

2. Section 24

24.1 Let $f_n(x) = \frac{1+2\cos^2 nx}{\sqrt{n}}$. Prove carefully that (f_n) converges uniformly to 0 on $\mathbb{R}$.

24.2 For $x \in [0, \infty)$, let $f_n(x) = \frac{x}{n}$.
(a) Find $f(x) = \lim f_n(x)$.
(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.
(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

24.4 Repeat Exercise 24.2 for $f(x) = \frac{x^n}{1+x^n}$

24.6 Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$.
(a) Does the sequence (f_n) converge pointwise on the set $[0, 1]$? If so, give the limit function.
(b) Does (f_n) converge uniformly on $[0, 1]$? Prove your assertion.