Homework 6

1. Section 24

24.9 Consider $f_n(x) = nx^n(1-x)$ for $x \in [0,1]$.
(a) Find $f(x) = \lim f_n(x)$. (b) Does $f_n \to f$ uniformly on $[0,1]$? Justify.

24.10 (a) Prove that if $f_n \to f$ uniformly, on a set S, and if $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.
(b) Do you believe the analogue of (a) holds for products? If so, see the next exercise.

24.13 Prove that if (f_n) is a sequence of uniformly continuous functions on an interval (a,b), and if $f_n \to f$ uniformly on (a,b), then f is also uniformly continuous on (a,b). *Hint*: Try an $\varepsilon/3$ argument as in the proof of Theorem 24.3.

24.14 Let $f_n(x) = \frac{nx}{1+n^2x^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$.
(a) Show $f_n \to f$ pointwise on $\mathbb{R}$.
(b) Does $f_n \to f$ uniformly on $[0,1]$? Justify.
(c) Does $f_n \to f$ uniformly on $[1,\infty)$? Justify.

2. Section 25

25.2 Let $f_n(x) = \frac{x^n}{n}$. Show (f_n) is uniformly convergent on $[-1,1]$ and specify the limit function.

25.4 Let (f_n) be a sequence of functions on a set $S \subset \mathbb{R}$, and suppose $f_n \to f$ uniformly on S. Prove (f_n) is uniformly Cauchy on S. *Hint*: Use the proof of Lemma 10.9 on page 63 as a model, but be careful.

25.5 Let (f_n) be a sequence of bounded functions on a set S, and suppose $f_n \to f$ uniformly on S. Prove f is a bounded function on S.

25.6 (a) Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.
(b) Does $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ represent a continuous function on $[-1,1]$?

25.10 (a) Show $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0,1)$.
(b) Show that the series converges uniformly on $[0,a]$ for each a, $0 < a < 1$.
(c) Does the series converge uniformly on $[0,1)$? Explain.