Homework 7

1. Section 26

26.2 (a) Observe $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$; see Example 1.
(b) Observe $\sum_{n=1}^{\infty} \frac{n}{2^n}$ Compare with Exercise 14.13(d) .
(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

26.6 Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.
(a) Prove $s' = c$ and $c' = -s$.
(b) Prove $(s^2 + c^2)' = 0$.
(c) Prove $s^2 + c^2 = 1$.
Actually $s(x) = \sin x$ and $c(x) = \cos x$, but you do not need these facts.

26.7 Let $f(x) = |x|$ for $x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all x? Discuss.

26.8 (a) Show $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$ for $x \in (-1, 1)$. *Hint:* $\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$. Let $y = -x^2$.
(b) Show $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1, 1)$.
(c) Show the equality in (b) also holds for $x = 1$. Use this to find a nice formula for π.
(d) What happens at $x = -1$?

2. Section 31

31.1 Find the Taylor series for $\cos x$ and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

31.4 Consider a, b in $\mathbb{R}$ where $a < b$. Show there exist infinitely differentiable functions f_a, g_b, $h_{a,b}$ and $h^*_{a,b}$ on $\mathbb{R}$ with the following properties. You may assume, without proof, that the sum, product, etc. of infinitely differentiable functions is again infinitely differentiable. The same applies to the quotient provided the denominator never vanishes.
(a) $f_a(x) = 0$ for $x \le a$ and $f_a(x) > 0$ for $x > a$. *Hint:* Let $f_a(x) = f(x - a)$ where f is the function in Example 3.
(b) $g_b(x) = 0$ for $x \ge b$ and $g_b(x) > 0$ for $x < b$.
(c) $h_{a,b}(x) > 0$ for $x \in (a, b)$ and $h_{a,b}(x) = 0$ for $x \notin (a, b)$.
(d) $h^*_{a,b}(x) = 0$ for $x \le a$ and $h^*_{a,b}(x) = 1$ for $x \ge b$. *Hint*: Use the function $\frac{f_a}{f_a + g_b}$.

31.5 Let $g(x) = e^{-1/x^2}$ for $x \ne 0$ and $g(0) = 0$.
(a) Show $g^{(n)}(0) = 0$ for all $n = 0, 1, 2, 3, \ldots$ *Hint*: Use Example 3.
(b) Show the Taylor series for g about 0 agrees with g only at $x = 0$.

3. Supplement Homework

S1 (a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all $x \in (-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.
(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the value of a_n.

S2 Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on $(-R, R)$, and assume $(x_n) \to 0$ with $x_n \ne 0$. If $g(x_n) = 0$ for all $n \in \mathbb{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

S3 Find an example of each of the following.

(a) An infinitely differentiable function $h(x)$ on all of $\mathbb{R}$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.

(b) An infinitely differentiable function $f(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.