Homework 8

1. Section 32

32.1 Find the upper and lower Darboux integrals for $f(x) = x^3$ on the interval $[0, b]$. *Hint*: Exercise 1.3 and Example 1 in §1 will be useful.

32.2 Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.
(a) Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.
(b) Is f integrable on $[0, b]$?

32.6 Let f be a bounded function on $[a, b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.

32.7 Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show g is integrable and $\int_a^b f = \int_a^b g$. *Hint:* First reduce to the case where f is the function identically equal to 0.

32.8 Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subset [a, b]$.

2. Section 33

33.3 A function f on $[a, b]$ is called a step function if there exists a partition $P = \{a = u_0 < u_1 < \cdots < u_m = b\}$ of $[a, b]$ such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for x in (u_{j-1}, u_j).
(a) Show that a step function f is integrable and evaluate $\int_a^b f$.
(b) Evaluate the integral $\int_0^4 P(x)dx$ for the postage-stamp function P in Exercise 17.16.

33.4 Give an example of a function f on $[0, 1]$ that is *not* integrable for which $|f|$ is integrable. *Hint*: Modify Example 2 in §32.

33.7 Let f be a bounded function on $[a, b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.
(a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions $P$of $[a, b]$. *Hint:* $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$.
(b) Show that if f is integrable on $[a, b]$, then f^2 also is integrable on $[a, b]$.

33.8 Let f and g be integrable functions on $[a, b]$.
(a) Show fg is integrable on $[a, b]$. *Hint*: Use Exercise 33.7 and $4fg = (f + g)2 - (f - g)2$.
(b) Show $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$. *Hint*: Exercise 17.8.

33.11 Let $f(x) = x\mathrm{sgn}(\sin \frac{1}{x})$for $x \neq 0$ and $f(0) = 0$.
(a) Show f is not piecewise continuous on $[-1, 1]$.
(b) Show f is not piecewise monotonic on $[-1, 1]$.
(c) Show f is integrable on $[-1, 1]$.