Homework 3
Due March 19, 2015

1. Some multilinear algebra

*Do this cleanly – don’t make a big mess of it, and don’t belabor easy stuff. Choose a basis when it makes the argument cleaner; don’t when it doesn’t.*

Let $V$ be an $n$-dimensional $\mathbb{R}$-vector space. Recall that

$$\Lambda V = \bigoplus_{k=0}^{n} \Lambda^k V$$

is the $\mathbb{R}$-algebra generated by $V$ subject to the relation

$$v \wedge w = -w \wedge v.$$

Thus we are regarding $\Lambda^k V$ as a quotient of $V \otimes \cdots \otimes V$, not as a subspace. Note that a general element of $\Lambda^k V$ is a sum of terms of the form $v_1 \wedge \cdots \wedge v_k$.

(a) Consider the pairing

$$\Lambda^k(V^*) \otimes \Lambda^k V \to \mathbb{R}$$

determined by

$$\langle \varphi_1 \wedge \cdots \wedge \varphi_k, v_1 \wedge \cdots \wedge v_k \rangle = \det \begin{pmatrix}
\varphi_1(v_1) & \cdots & \varphi_1(v_k) \\
\vdots & \ddots & \vdots \\
\varphi_k(v_1) & \cdots & \varphi_k(v_k)
\end{pmatrix}.$$

Say briefly why this is well-defined. Show that it is non-degenerate; that is, for every $\omega \in \Lambda^k(V^*)$ there is an $X \in \Lambda^k V$ such that $\langle \omega, X \rangle \neq 0$. Thus the map

$$\Lambda^k(V^*) \to (\Lambda^k V)^*$$

$$\omega \mapsto \langle \omega, - \rangle$$

is an isomorphism.
(b) For \( l \leq k \), define contraction
\[
\mathcal{D} : \Lambda^l V \otimes \Lambda^k V^* \to \Lambda^{k-l} V^*
\]
as the adjoint of wedging
\[
\wedge : \Lambda^l V \otimes \Lambda^{k-l} V \to \Lambda^k V.
\]
That is, for \( X \in \Lambda^l V \) and \( Y \in \Lambda^{l-k} V \) and \( \omega \in \Lambda^k V^* \) we have
\[
\langle X \mathcal{D} \omega, Y \rangle = \langle \omega, X \wedge Y \rangle.
\]
Given \( v, w \in V \) and \( \varphi_1, \ldots, \varphi_k \in V^* \), check that
\[
v \mathcal{D} (\varphi_1 \wedge \cdots \wedge \varphi_k) \quad \text{and} \quad (v \wedge w) \mathcal{D} (\varphi_1 \wedge \cdots \wedge \varphi_k)
\]
agree with what you think they should be.

(c) Fix a non-zero \( \text{vol} \in \Lambda^n V^* \). Show that map
\[
\Lambda^k V \to \Lambda^{n-k} V^*
\]
\[
X \mapsto X \mathcal{D} \text{vol}
\]
is an isomorphism.

(d) Fix an inner product \( g \) on \( V \). This determines an isomorphism
\[
V \xrightarrow{\tilde{g}} V^*
\]
\[
v \mapsto g(v, -)
\]
and thus an isomorphism
\[
\Lambda^k V \xrightarrow{\tilde{g}} \Lambda^k V^*
\]
\[
v_1 \wedge \cdots \wedge v_k \mapsto g(v_1, -) \wedge \cdots \wedge g(v_k, -)
\]
for each \( k \). Combining this isomorphism with the pairing of part (a) we get pairings on \( \Lambda^k V \) and \( \Lambda^k V^* \), which we will also call \( g \). Check that these are inner products (symmetric and positive definite).

(e) For \( X \in \Lambda^k V \), consider both
\[
g(X, X) \cdot \text{vol} \quad \text{and} \quad (X \mathcal{D} \text{vol}) \wedge \tilde{g}(X)
\]
in \( \Lambda^n V^* \). How do they compare? Given \( g \), is there a best choice for \( \text{vol} \)?
Optional: If you have anything else to say about these operations on $\Lambda V$ and $\Lambda V^*$, say it.

2. Connections as splittings

(a) Let $\pi: E \to M$ be a vector bundle. There is a canonical section of the vector bundle $\pi^*E$ on $E$. What is it?

(b) Recall that a connection on $E$ determines a splitting of the natural exact sequence

$$0 \to \pi^*E \to TE \xrightarrow{D\pi} \pi^*TM \to 0$$

of vector bundles on $E$. Take $E = TM$, so this becomes

$$0 \to \pi^*TM \to TTM \xrightarrow{D\pi} \pi^*TM \to 0.$$  

With a splitting of this sequence we can take the canonical section of the right-hand $\pi^*TM$ and get a section of the middle term $TTM$, i.e. a vector field on $TM$. Show that this is the geodesic spray: that is, if $\gamma: (0,1) \to M$ is a curve and $\tilde{\gamma}: (0,1) \to TM$ is the lift discussed in class, then $\gamma$ is a geodesic if and only if $\tilde{\gamma}$ is an integral curve of this vector field on $TM$.

(c) Optional: Explore what it means for a connection on $TM$ to be symmetric, in terms of the splitting of

$$0 \to \pi^*TM \to TTM \xrightarrow{D\pi} \pi^*TM \to 0.$$