FIELD GUIDE TO $A_\infty$ SIGN CONVENTIONS

1. SIGNS IN THE DEFINITION OF $A_\infty$-ALGEBRA

We are concerned with the following “standard” definition. An $A_\infty$-algebra is a $\mathbb{Z}$-graded vector space $A$ equipped with linear maps $m_n$ of degree $2 - n$, for $n \geq 1$, satisfying for each $n$ the identity

$$\sum_{k+l=n+1} \sum_{k=1}^k \epsilon \cdot (-1)^{l(\bar{a}_1 + \ldots + a_{j+1})} \cdot m_k(a_1, \ldots, a_{j-1}, m_l(a_j, \ldots, a_{j+l-1}), a_{j+l}, \ldots, a_n),$$

where $\bar{a}$ is the degree of a homogeneous element $a \in A$, $\epsilon = \epsilon(k, l, j)$ is a certain sign. The second sign in the definition is obtained from the Koszul sign rule, since $m_l$, that has degree $2 - l$, is exchanged with the elements $a_1, \ldots, a_{j-1}$. Note that Seidel and Fukaya use another format of the definition where the Koszul sign is not inserted (see below).

The problem is that the sign $\epsilon(k, l, j)$ in different sources is written differently. The works [2] and [4] use

$$\epsilon_1(k, l, j) = (-1)^{r+st},$$

where $r = j - 1$, $s = l$ and $t = n - j - l + 1$. The original work where $A_\infty$-algebra was introduced, [10], uses

$$\epsilon_2(k, l, j) = (-1)^{j(l+1)+ln}.$$

[6] and [3] (and following them, [7]) use

$$\epsilon_3(k, l, j) = (-1)^{(j-1)(l-1)+(k-1)l} = (-1)^{j(l+1)+kl+1}.$$

[1] (and following it, [8]) uses

$$\epsilon_4(k, l, j) = (-1)^{j(l+1)}.$$

As one can easily check, the connection between the first three signs is the following:

$$\epsilon_1 = \epsilon_3 = -\epsilon_2.$$

In particular, these signs all give the same definition of an $A_\infty$-algebra. This definition corresponds to associating to $(m_n)$ a coderivation $D$ of the free coalgebra $T(A[1])$ cogenerated by $A[1]$, and setting $D^2 = 0$.

The relation with $\epsilon_4$ is more complicated:

$$\epsilon_4 = (-1)^{\binom{n+1}{2}+1+\binom{k}{2}+\binom{l}{2}} \epsilon_1.$$

This means that in order to match the $\epsilon_4$-definition of an $A_\infty$-algebra with the $\epsilon_1$-definition one has to change the $m_n$ as follows:

$$m'_n = (-1)^{\binom{2}{2}} m_n.$$
Seidel uses very different sign conventions. In [9] the $A_{\infty}$-identity looks as follows:

$$\sum_{k+l=n+1} \sum_{k=1}^k (-1)^{a_1+\ldots+a_{j-1}+j-1} \cdot m_k(a_1,\ldots,a_{j-1},m_l(a_j,\ldots,a_{j+l-1}),a_{j+l},\ldots,a_n),$$

which for example does not correspond to the usual associativity if there is only $m_2$. To connect this to the definition corresponding to $\epsilon_1$ one has to make the following change of sign:

$$m'_n(a_1,\ldots,a_n) = (-1)^{(n-1)a_1+(n-2)a_2+\ldots+a_{n-1}} m_n(a_1,\ldots,a_n).$$

2. Sign in the definition of the Gerstenhaber bracket

Interpreting a Hochschild cochain $f \in \text{Hom}(A^\otimes n,A)$ as giving rise to a coderivation $D_f$ of the free coalgebra $T(A[1])$, we obtain the definition of Gerstenhaber bracket $[f,g]$ of Hochschild cochains, so that

$$D_{[f,g]} = [D_f,D_g],$$

where on the right we have a supercommutator. This leads to the following formula for $f \in \text{Hom}(A^\otimes m,A)$, $g \in \text{Hom}(A^\otimes n,A)$

$$[f,g] = f \circ g - (-1)^{|f||g|} g \circ f, \quad \text{where} \quad f \circ g(a_1,\ldots,a_{m+n-1}) = \sum_{i=1}^m (-1)^{(a_1+\ldots+a_{i-1}+m-1) \deg(g)+(i-1)(n-1)} f(a_1,\ldots,a_{i-1},g(a_i,\ldots,a_{i+n-1}),a_{i+n},\ldots,a_{m+n-1}),$$

where for a cochain $f \in \text{Hom}(A^\otimes m,A)$, homogeneous of degree $\deg(f)$, we set $|f| = \deg(f) + m - 1$. Note that in the case when $A$ sits in degree 0 this is compatible with the usual definition, say, in [5, E.1.5.2].

Bibliography