THETA IDENTITIES WITH COMPLEX MULTIPLICATION

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Introduction. This paper grew out from the attempt to refine the notion of a symmetric line bundle on an abelian variety in the case of complex multiplication. Recall that a line bundle $L$ on an abelian variety $A$ is called symmetric if $(-\text{id}_A)^*L \cong L$. It is known that in this case one has an isomorphism

$$(n \text{id}_A)^*L \cong L^n$$

for any $n \in \mathbb{Z}$. Now assume that $A$ admits a complex multiplication by a ring $R$, that is, we have a ring homomorphism $R \to \text{End}(A) : r \mapsto [r]_A$. If $L$ is non-degenerate, then the corresponding polarization $\phi_L : A \to \hat{A}$ (where $\hat{A}$ is the dual abelian variety to $A$) defines the Rosati involution on $\text{End}(A) \otimes \mathbb{Q}$ (see [5]). Assume that this involution is compatible with some involution $\varepsilon$ on $R$. Let $R^+ \subset R$ be the subring of $\varepsilon$-invariant elements. Then for every $r \in R^+$, the homomorphism $\phi_L \circ [r]_A : A \to \hat{A}$ is self-dual; hence, one can ask whether it comes from some "natural" line bundle $L(r)$ on $A$. The word "natural" should mean in particular that the map $r \mapsto L(r)$ from $R^+$ to the group of symmetric line bundles on $A$ is a homomorphism, resembling the usual homomorphism $n \mapsto L^n$. By analogy with the above isomorphism, we would like to impose the following condition on such a homomorphism

$$[r]_A^*L(r_0) \cong L(\varepsilon(r)r_0r)$$

for any $r \in R$, $r_0 \in R^+$. We call such data a $\Sigma_{R,\varepsilon}$-structure (since a suitable generalization of this notion to group schemes with complex multiplication is a refinement of the notion of $\Sigma$-structure defined by L. Breen in [2]).

In the first part of the paper we describe an obstruction to the existence of a $\Sigma_{R,\varepsilon}$-structure for a given polarization of $A$. It turns out that when $R$ is commutative, one can prove the existence of a $\Sigma_{R,\varepsilon}$-structure, assuming that $R$ is unramified at all $\varepsilon$-stable places above 2 (in noncommutative cases, one also needs some additional assumptions at archimedean places). In the case of an elliptic curve $E$ with its standard principal polarization and $R = \text{End}(E)$ this result is sharp: a $\Sigma_{R,\varepsilon}$-structure exists if and only if $R$ is unramified at 2. In the case of commutative real multiplication, one needs only that $R$ is normal above 2 to ensure the existence of a $\Sigma_{R,\varepsilon}$-structure.

Received 31 March 1997. Revision received 16 September 1997.
1991 Mathematics Subject Classification. Primary 14K25.
In the second part of the paper, we establish an analogue of generalized Riemann's theta relations (see, e.g., [6]) for theta functions with complex multiplication. Instead of an integer-valued matrix $B$ such that $B^t \cdot B = n \cdot \text{Id}$, where $n \in \mathbb{Z}_{>0}$, our identity uses a matrix $A$ with elements in $R$ (where the abelian variety in question has a complex multiplication by $R$) such that $B^\varepsilon \cdot B = n \cdot \text{Id}$, where $B^\varepsilon$ is obtained by applying $\varepsilon$ to all entries of $B^t$. The existence of a $\Sigma_R,\varepsilon$-structure is reflected in the simplification of the expression for theta-characteristics in the right-hand side of this identity (see (2.3.6)).

In [7] we interpret the notion of a symmetric cube structure ($\Sigma$-structure in the terminology of [2]) as a monoidal functor from the category of integer-valued symmetric forms to the category of abelian varieties equipped with line bundles. The notion of $\Sigma_R,\varepsilon$-structure arises when one tries to find a similar picture in the case of complex multiplication. In the present paper we show (Theorem 1.3.2) that a $\Sigma_R,\varepsilon$-structure indeed leads to a monoidal functor from the category of $\varepsilon$-hermitian, projective $R$-modules. The results of Section 1.5 on the existence of $\Sigma_R,\varepsilon$-structure and the simplest example of theta-identity with complex multiplication can also be found in [7].

Acknowledgment. I am grateful to B. Gross for helpful discussions.

1. Line bundles on abelian varieties with complex multiplication

1.1. Basic operations on abelian varieties with complex multiplication. Let $R$ be a ring and $A$ be an abelian variety with complex multiplication by $R$; that is, a homomorphism $R \to \text{End}(A)$ is given. For an element $r \in R$ we denote by $[r]_A$ the corresponding endomorphism of $A$.

Given a finitely generated, projective right $R$-module $P$, one can define the tensor product $P \otimes_R A$ (which is an abelian variety) based on the property

$$\text{Hom}(P \otimes_R A, A') \simeq \text{Hom}_R(P, \text{Hom}(A, A'))$$  \hspace{1cm} (1.1.1)$$

for any abelian variety $A'$, where the left $R$-action on $A$ induces the right $R$-action on $\text{Hom}(A, A')$. Notice that when the ring $R$ is commutative, there is a natural $R$-action on the tensor product $P \otimes_R A$ defined above. In particular, when $R$ is commutative, tensoring with rank-1 projective $R$-modules $P$ gives the well-known action of the group $\text{Pic}(R)$ on the set of abelian varieties with complex multiplication by $R$.

Similarly, if $Q$ is a finitely generated, projective left $R$-module and $A$ has complex multiplication by $R$, then one can define an abelian variety $\text{Hom}_R(Q, A)$ such that

$$\text{Hom}(A', \text{Hom}_R(Q, A)) \simeq \text{Hom}_R(Q, \text{Hom}(A', A))$$  \hspace{1cm} (1.1.2)$$

for any abelian variety $A'$, where $\text{Hom}(A', A)$ is equipped with the natural left
It is easy to see that

$$\text{Hom}_R(Q, A) \simeq \text{Hom}_R(Q, R) \otimes_R A,$$

where $\text{Hom}_R(Q, R)$ is considered as a right $R$-module in the natural way.

For an abelian variety $A$, we denote by $\hat{A}$ the dual abelian variety. If $A$ has complex multiplication by $R$, then the dual variety $\hat{A}$ has the induced complex multiplication by the opposite ring $R^{\text{op}}$, such that $[r]_A = [r]_{\hat{A}}$. For any finitely generated, projective right $R$-module $P$, one has a canonical isomorphism

$$P \otimes_R A \simeq \text{Hom}_{R^{\text{op}}}(P, \hat{A}),$$

(1.1.3)

where in the right-hand side $P$ is considered as a left $R^{\text{op}}$-module.

Now assume that $R$ is equipped with an involution $\epsilon: R \to R$; that is, $\epsilon$ is an antiautomorphism of $R$ such that $\epsilon^2 = \text{id}$. Then we can convert the complex multiplication by $R^{\text{op}}$ on $\hat{A}$ into a complex multiplication by $R$ using $\epsilon$. Hence, the isomorphism (1.1.3) can be rewritten as

$$P \otimes_R A \simeq \text{Hom}_{R^{\text{op}}}(P^\epsilon, \hat{A}) \simeq \text{Hom}_R(P^\epsilon, R) \otimes_R \hat{A},$$

(1.1.4)

where $P^\epsilon$ is the left $R$-module obtained from $P$ using the involution $\epsilon$.

1.2. Sesquilinear forms and biextensions. There is a bijective correspondence between homomorphisms of abelian varieties $A_2 \to A_1$ and biextensions of $A_1 \times A_2$ by $G_m$. Recall that the latter are given by line bundles $\mathcal{B}$ on $A_1 \times A_2$ together with isomorphisms

$$(p_1 + p_2, p_3)^* \mathcal{B} \simeq p_{13}^* \mathcal{B} \otimes p_{23}^* \mathcal{B},$$

$$(p_1, p_2 + p_3)^* \mathcal{B} \simeq p_{12}^* \mathcal{B} \otimes p_{13}^* \mathcal{B}$$

on $A_1 \times A_1 \times A_2$ and $A_1 \times A_2 \times A_2$, satisfying some natural compatibility conditions (see [2]). For a homomorphism $\phi: A_2 \to \hat{A}_1$, the corresponding biextension $\mathcal{B}_\phi$ is given by a line bundle $(\text{id}, \phi)^* \mathcal{P}$ on $A_1 \times A_2$, where $\mathcal{P}$ is the Poincaré line bundle on $A_1 \times \hat{A}_1$.

If $A_1$ and $A_2$ have complex multiplications by $R^{\text{op}}$ and $R$, respectively, then the condition that a homomorphism $A_2 \to \hat{A}_1$ is compatible with $R$-action is equivalent to the condition that the corresponding biextension $\mathcal{B}$ of $A_1 \times A_2$ is equipped with natural isomorphisms

$$a_r : (r \times \text{id})^* \mathcal{B} \simeq (\text{id} \times r)^* \mathcal{B}$$

for every $r \in R$. If we write the $R^{\text{op}}$-action on $A_1$ as the right $R$-action, then isomorphisms $a_r$ can be written symbolically as $\mathcal{B}_{x, y} \simeq \mathcal{B}_{x, ry}$. These isomorphisms
are compatible with the structure of biextension on $\mathcal{B}$ and with the $R$-module structure on $A$ as explained in the following definition (cf. [4, VII 2.10.3], where the case of the commutative ring $R$ is considered).

**Definition 1.2.1.** An $R$-biextension of $A_1 \times A_2$ is a biextension $\mathcal{B}$ of $A_1 \times A_2$ together with a system of isomorphisms of biextensions $a_r$ as above, such that

1. the composition

   $$\mathcal{B}_{x, (r+r')y} = \mathcal{B}_{x, r+y} \otimes \mathcal{B}_{x, r'y} \xrightarrow{a_r \otimes a_{r'}} \mathcal{B}_{x, r+y+r'y}$$

   where $c$ is the isomorphism giving a structure of biextension on $\mathcal{B}$, coincides with $a_{r+r'}$;

2. the composition

   $$\mathcal{B}_{x, r+y} \xrightarrow{a_r} \mathcal{B}_{x, r'y} \xrightarrow{a_{r'}} \mathcal{B}_{x, r'y}$$

   coincides with $a_{r'}$.

It is easy to see that $R$-biextensions of $A_1 \times A_2$ correspond bijectively to homomorphisms $A_1 \rightarrow A_2$ compatible with $R$-action. However, the definition above has an advantage in that it can be given for arbitrary group schemes.

Now if $R$ is equipped with involution $\varepsilon$, and if $A_1, A_2$ are abelian varieties with complex multiplication by $R$, then we can define an $(R, \varepsilon)$-biextension of $A_1 \times A_2$ to be an $R$-biextension of $A_1^\varepsilon \times A_2$, where $A_1^\varepsilon$ is $A_1$ with a complex multiplication by $R^\text{op}$ induced by $\varepsilon$. In other words, an $(R, \varepsilon)$-biextension is a biextension $\mathcal{B}$ of $A_1 \times A_2$ together with isomorphisms $\mathcal{B}_{x, r+y} \simeq \mathcal{B}_{x, r'y}$ for $r \in R$, satisfying the compatibility conditions analogous to the conditions (1) and (2) in Definition 1.2.1. The corresponding homomorphism $\phi : A_2 \rightarrow A_1$ satisfies $\phi \circ [r]_{A_2} = [\varepsilon(r)]_{A_1} \circ \phi$ for all $r \in R$. If $\phi$ is an isogeny, then this is equivalent to the following equality in $\text{End}(A_2) \otimes \mathbb{Q}$:

$$[\varepsilon(r)]_{A_2} = \phi^{-1} \circ [r]_{A_1} \circ \phi.$$
we have a natural isomorphism \( \langle g^* L_1, L_2 \rangle \simeq \langle L_1, (g^{-1})^* L_2 \rangle \), where \( \langle \cdot, \cdot \rangle \) denotes the symbol defined in [3].

Let \( P_1 \) be a left \( R \)-module and \( P_2 \) a right \( R \)-module. A sesquilinear form \( b : P_1 \times P_2 \to R \) is a \( \mathbb{Z} \)-bilinear map such that \( b(rx, y) = rb(x, y) \), \( b(x, yr) = b(x, y)r \). This is the same as a morphism of left \( R \)-modules \( P_1 \to \text{Hom}_R(P_2, R) \) or a morphism of right \( R \)-modules \( P_2 \to \text{Hom}_R(P_1, R) \). Note that if \( R \) is equipped with an involution \( \varepsilon \), then we can convert right \( R \)-modules into left ones, and vice versa. Thus, if \( P'_1 \) and \( P_2 \) are right \( R \)-modules, then \( b : P'_1 \times P_2 \to R \) is a sesquilinear form if \( b \) is \( \mathbb{Z} \)-bilinear, and \( b(xr, y) = \varepsilon(r)b(x, y) \), \( b(x, yr) = b(x, y)r \) for every \( r \in R \).

Let \( A_1 \) be an abelian variety with complex multiplication by \( R^\text{op} \), and let \( A_2 \) be an abelian variety with complex multiplication by \( R \). Assume we are given a homomorphism \( : A_2 \to A_1 \) compatible with \( R \)-action. Then for every sesquilinear form \( b : P_1 \times P_2 \to R \), one can construct a canonical homomorphism of abelian varieties

\[
\phi(b) : (P_2 \otimes_R A_2) \to P_1 \otimes_{R^\text{op}} A_1.
\]

Namely, using (1.1.3), (1.1.2), and (1.1.1), we can write

\[
\text{Hom}(P_2 \otimes_R A_2, P_1 \otimes_{R^\text{op}} A_1) \simeq \text{Hom}(P_2 \otimes_R A_2, \text{Hom}_R(P_1, A_1))
\]

\[
\simeq \text{Hom}_R(P_1, \text{Hom}(P_2 \otimes_R A_2, A_1))
\]

\[
\simeq \text{Hom}_R(P_1, \text{Hom}_R(P_2, \text{Hom}(A_2, A_1))).
\]

Now we can produce an element in the latter group, that is, a homomorphism of left \( R \)-modules \( P_1 \to \text{Hom}_R(P_2, \text{Hom}(A_2, A_1)) \) by the formula \( x \mapsto (y \mapsto \phi \circ [b(x, y)]_{A_2}) \).

Thus, every \( R \)-bisetension \( \mathcal{B} \) of \( A_1 \times A_2 \) induces a map \( b \mapsto \mathcal{B}(b) \) from the set of sesquilinear forms \( b : P_1 \times P_2 \to R \) to biextensions of \( (P_1 \otimes_{R^\text{op}} A_1) \times (P_2 \otimes_R A_2) \). The original biextension \( \mathcal{B} \) is obtained as \( \mathcal{B}(b_1) \) for \( P_1 = R \) as a left \( R \)-module, and for \( P_2 = R \) as a right \( R \)-module, \( b_1(r_1, r_2) = r_1 r_2 \). One can easily see that

\[
\mathcal{B}(b_1 + b_2) \simeq \mathcal{B}(b_1) \otimes \mathcal{B}(b_2).
\]  

(1.2.1)

Also if \( f_1 : P_1 \to P'_1 \) and \( P_2 \to P'_2 \) are morphisms of \( R \)-modules and \( b' : P'_1 \times P'_2 \to R \) is a sesquilinear form, then \( b = (f_1, f_2)^* b' : P_1 \times P_2 \to R \) is sesquilinear and \( \mathcal{B}(b) \simeq (f_1 \otimes_R A \times f_2 \otimes_R A)^* \mathcal{B}(b') \). For example, for every \( r \in R \) we have a morphism of right \( R \)-modules \( l(r) : R \to R : r' \mapsto rr' \). Then the pull-back of the form \( b_1 \) by \( (\text{id}, l(r)) \) is the sesquilinear form \( b_r(r_1, r_2) = r_1 r_2 \). The above functoriality implies that

\[
\mathcal{B}(b_r) \simeq (\text{id} \times [r]_{A_2})^* \mathcal{B}.
\]  

(1.2.2)
Note that we can consider \( P_2 \) as a left \( R^{\text{op}} \)-module and \( P_1 \) as a right \( R^{\text{op}} \)-module. Then \( b \) induces a sesquilinear form \( b^{\text{op}} : P_2 \times P_1 \to R^{\text{op}} \) on these \( R^{\text{op}} \)-modules. Now the biextension \( \mathcal{B}(b^{\text{op}}) \) of \( A_2 \times A_1 \) is obtained from \( \mathcal{B}(b) \) by permutation of factors.

When \( R \) is equipped with an involution \( \epsilon \), one can identify \( R^{\text{op}} \) with \( R \) and rewrite the above constructions using only right \( R \)-modules.

1.3. Hermitian forms and line bundles. Let \( A \) be an abelian variety with complex multiplication by \( R \) and \( \epsilon \) be an involution on \( R \). Recall that every (rigidified) line bundle \( M \) on \( A \) defines a symmetric biextension \( \Lambda(M) \) of \( A^2 \) by the formula

\[
\Lambda(M) = m^* M \otimes p_1^* M^{-1} \otimes p_2^* M^{-1},
\]

which corresponds to a symmetric morphism \( \phi_M : A \to \hat{A} \). Now assume that \( \Lambda(M) \) is an \((R, \epsilon)\)-biextension. Then for every finitely generated, projective right \( R \)-module \( P \) and a sesquilinear form \( b : P \times P \to R \), the construction of the previous section gives a biextension \( \mathcal{B}(b) \) of \( (P \otimes_R A)^2 \). It is easy to see that this biextension is symmetric provided that \( b \) is a hermitian form; that is, \( b \), in addition to being sesquilinear, satisfies the identity \( b(y, x) = \epsilon(b(x, y)) \). We are going to study the following question: When for every hermitian form \( b \) can one find a “natural” line bundle \( L(b) \) on \( P \otimes_R A \) such that \( \mathcal{B}(b) \simeq \Lambda(L(b)) \)?

"Natural" means that if \( b = f^* b' \) for some morphism of \( R \)-modules \( f : P \to P' \), then \( L(b) = (f \otimes_R A)^* L(b') \), and if \( (P, b) = (P_1, b_1) \oplus (P_2, b_2) \) is a direct sum in the category of hermitian modules, then \( L(b) \) is the external tensor product of \( L(b_1) \) and \( L(b_2) \). To see what this means, note that for any \( r \in R^+ = \{ r_1 \in R \mid \epsilon(r_1) = r_1 \} \), we have the hermitian form \( h_r \) on \( R \) defined by \( h_r(1, 1) = r \); that is, \( h_r(x, y) = \epsilon(x)ry \). Thus, we should have the set of line bundles \( L(r) \) on \( A \) corresponding to the forms \( h_r \). The “naturality” imposes certain restrictions on \( L(r) \), which are described in the following definition.

**Definition 1.3.1.** Let \( A \) be an abelian variety, \( R \to \text{End}(A) : r \mapsto [r]_A \) a ring homomorphism, and \( \phi : A \to \hat{A} \) a symmetric homomorphism (that is, \( \hat{\phi} = \phi \)) such that \( \phi \circ [\epsilon(r)]_A = [r]_A \circ \phi \) for any \( r \in R \), where \([r]_A = [r]_A \). Then a \( \Sigma_{R, \epsilon} \)-structure for \( \phi \) is a homomorphism \( R^+ \to \text{Pic}^+(A) : r_0 \mapsto L(r_0) \), where \( \text{Pic}^+(A) \) is the group of symmetric line bundles on \( A \) such that

\[
\phi_{L(r_0)} = \phi \circ [r_0]_A \tag{1.3.1}
\]

for any \( r_0 \in R^+ \) and

\[
r^* L(r_0) \simeq L(\epsilon(r)r_0) \tag{1.3.2}
\]

for any \( r \in R, r_0 \in R^+ \).
Note that (1.3.1) and (1.3.2) lead to the isomorphism

$$L(\varepsilon(r) + r) \simeq ([r], \phi)^{\mathcal{P}} \tag{1.3.3}$$

for any $r \in R$. (Apply (1.3.2) to $r$ and $r + 1$ and use an isomorphism $\Lambda(L(1)) \simeq (\text{id} \times \phi)^{\mathcal{P}}$ on $A \times A$.) If $L(r)$ is a $\Sigma_{R,\varepsilon}$-structure for $\phi$, then any other $\Sigma_{R,\varepsilon}$-structure for $\phi$ has the form

$$L'(r) = L(r) \otimes \eta(r),$$

where $\eta : R^+ \to \text{Pic}^+(A)$ is a homomorphism such that $r^*\eta(r_0) \simeq \eta(\varepsilon(r)r_0r)$ for any $r \in R$, $r_0 \in R^+$. It follows from (1.3.3) that for such $\eta$ we also have $\eta(\varepsilon(r) + r) = 0$.

There is a trivial example of $\Sigma_{R,\varepsilon}$-structure for $2\phi$: $L(r) = (\text{id}, \phi \circ [r]_A)^{\mathcal{P}}$, where $\mathcal{P}$ is the Poincaré line bundle on $A \times A$. In particular, if $\phi = \phi_M$ for a symmetric line bundle $M$ on $A$, then $L(1) \simeq M^2$ in this example. The natural question is under what condition on $M$ there exists a $\Sigma_{R,\varepsilon}$-structure with $L(1) = M$. Below we consider this question for symmetric line bundles of degree 1 on elliptic curves. Now we are going to show that a $\Sigma_{R,\varepsilon}$-structure induces a monoidal functor from the category of hermitian forms to the category of line bundles over abelian varieties.

**Theorem 1.3.2.** Assume that a $\Sigma_{R,\varepsilon}$-structure $L(\cdot) : R^+ \to \text{Pic}^+(A)$ for $\phi$ is given. Then for every finitely generated, projective right $R$-module $P$ and a hermitian form $h$ on $P$, there is a canonical symmetric line bundle $L(h)$ on $P \otimes_R A$ such that $A(L(h)) \simeq (h)$. Furthermore, if $f : P \to P'$ is a morphism of such modules and $h = f^*h'$, then $L(h) \simeq (f \otimes_R A)^*L(h')$. Also, if $(P, h) \simeq (P_1, h_1) \oplus (P_2, h_2)$, then $L(h)$ is isomorphic to the external tensor product of $L(h_1)$ and $L(h_2)$.

**Proof.** For every collection of elements $x_1, \ldots, x_n \in P$, we denote by $i_{x_1, \ldots, x_n} : R^n \to P$ the corresponding morphism of right $R$-modules: $i_{x_1, \ldots, x_n}(r_1, \ldots, r_n) = x_1r_1 + \cdots + x_nr_n$. We define $L(h)$ as a unique rigidified line bundle on $P \otimes_R A$ such that for every element $x \in P$, one has

$$(i_x \otimes_R A)^*L(h) \simeq L(h(x, x)),$$

and for every pair of elements $x_1, x_2 \in P$, one has

$$(i_{x_1, x_2} \otimes_R A)^*L(h) \simeq p_1^*L(h(x_1, x_1)) \otimes p_2^*L(h(x_2, x_2)) \otimes (\text{id} \times \phi \circ [h(x, y)]_A)^{\mathcal{P}},$$

where we identify $R^2 \otimes_R A$ with $A^2$, $p_i$, $i = 1, 2$ are the projections of $A^2$ on $A$, and $\mathcal{P}$ is the Poincaré bundle. First, let us check the uniqueness. When $P = R^n$ the uniqueness follows immediately from the theorem of cube. For arbitrary $P$ we can choose a surjective morphism $f : R^n \to P$. Then it follows by definition that $(f \otimes_R A)^*L(h) = L(f^*h)$, where $f^*h$ is the induced form on $R^n$. Since $f$ is a
projection onto a direct summand this implies the uniqueness of \( L(h) \). As for existence, let us begin with the case \( P = \mathbb{R}^n \). Then if \( \{e_1, \ldots, e_n\} \) is the standard base of \( \mathbb{R}^n \), let us denote \( h_{ij} = h(e_i, e_j) \) and set

\[
L(h) = \bigotimes_i p_i^* L(h_{ii}) \otimes \bigotimes_{i<j} p_{ij}^* \langle [h_{ij}]_A, \phi \rangle^* \mathcal{P}.
\]

One can check easily that the required isomorphisms hold. Now to prove the existence of \( L(h) \) in general, choose a surjection \( f : \mathbb{R}^n \to P \). Then it is sufficient to check that \( L(f^*h) \) is in fact a pull-back of some line bundle on \( P \otimes \mathbb{R} A \) by \( f : \mathbb{R} \otimes \mathbb{R} A : \mathbb{R}^n \to P \otimes \mathbb{R} A \). In other words, we have to check that two pull-backs of \( L(f^*h) \) to the fiber product \( A^n \times P \otimes \mathbb{R} A \) are the same. But this fiber product is of the form \( Q \otimes \mathbb{R} A \), where \( Q = \ker(\mathbb{R}^n \otimes \mathbb{R}^n (f^{-1} f) \to \mathbb{R}^n) \) and two projections to \( A^n \) are induced by the natural projections \( g_1, g_2 : Q \to \mathbb{R}^n \). Now the required isomorphism of two pull-backs of \( L(f^*h) \) follows from the equality \( g_1^* f^* h = g_2^* f^* h \) of hermitian forms on \( Q \). This proves the existence of \( L(h) \). In the case \( P = \mathbb{R}^n \), using (1.2.1) and (1.2.2) one easily shows that \( \Lambda(L(h)) = \mathcal{A}_\phi(h) \). The case of general \( P \) follows by considering a surjection \( \mathbb{R}^n \to P \) as before. The functoriality of \( L(h) \) in \( h \) follows from its construction.

1.4. Case of elliptic curve. Let us consider the case when \( A = E \) is an elliptic curve, \( \phi_0 : E \cong \hat{E} \) is the standard principal polarization induced by the line bundle \( \mathcal{O}(e) \), where \( e \in E \) is the neutral element and \( R \subset \text{End}(E) \) is a subring closed under the Rosati involution. We assume that the ground field \( k \) is algebraically closed and \( \text{char}(k) \neq 2 \). It is known that \( R^+ \subset \mathbb{Z} \); hence, a \( \Sigma_{R, e} \)-structure for \( \phi_0 \) is determined uniquely by the line bundle \( L(1) \), which should be of the form \( \mathcal{O}(p) \) where \( p \in E_2 \) is a point of order 2 on \( E \).

**Proposition 1.4.1.** Fix a point \( p \in E_2 \). The following conditions are equivalent:

1. there exists a \( \Sigma_{R, e} \)-structure for \( \phi_0 \) with \( L(1) = \mathcal{O}(p) \);
2. for every \( r \in R \) such that \( r|_{E_2} \neq 0 \), one has either \( p \notin r(E_2) \), or \( r(E_2) = E_2 \) and \( r(p) = p \).

**Proof.** The line bundle \( L(1) = \mathcal{O}(p) \) defines a \( \Sigma_{R, e} \)-structure if and only if for every \( r \in R, r \neq 0 \) there is an isomorphism

\[
\mathcal{O}(N(r)p) \cong r^* L(1) = \mathcal{O}(r^{-1}(p)),
\]

where \( N(r) = e(r)r \in \mathbb{Z} \). Since the divisor \( r^{-1}(p) \subset E \) is symmetric, this is equivalent to the following equality in \( E \):

\[
\sum_{x \in r^{-1}(p) \cap E_2} x = N(r)p. \tag{1.4.1}
\]
Note that \( N(r) = \deg(r) \equiv |\ker(r|_{E_2})| \mod(2) \). Thus, \( N(r)p = 0 \) if and only if \( \ker(r|_{E_2}) \neq 0 \), otherwise \( N(r)p = p \). In particular, both parts of (1.4.1) are equal to zero when \( p \neq r(E_2) \). Now assume that \( p = r(x_0) \). If, in addition, \( r|_{E_2} \) is invertible, then (1.4.1) becomes \( x_0 = p \); that is, \( r(p) = p \). Otherwise, \(|\ker(r|_{E_2})| = 2 \) and (1.4.1) becomes \( \sum_{x \in \ker(r|_{E_2})} x = 0 \), which is impossible.

In the case \( p = e \), the above proposition implies that a \( \Sigma_{R,e} \)-structure with \( L(1) = \mathcal{O}(e) \) exists if and only if for every \( r \in R \) the restriction \( r|_{E_2} \) is either zero or invertible; that is, the image of \( R \) under the natural homomorphism \( \text{End}(E) \to \text{End}(E_2) \) is a field. Note that there is a maximal subfield \( \mathbb{F}_4 \) in the matrix algebra \( M_2(\mathbb{F}_2) \). Namely, \( \mathbb{F}_4 = \{0, I, A, A^2 = A + I\} \), where \( I \) is the identity matrix \( A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \). This means that the maximal subalgebra \( R_0 \) of \( \text{End}(E) \), for which there exists a \( \Sigma_{R_0,e} \)-structure with \( L(1) = \mathcal{O}(e) \), is the preimage of \( \mathbb{F}_4 \) under the homomorphism \( \text{End}(E) \to \mathbb{M}(2,\mathbb{F}_2) \). For example, if \( \text{End}(E) \) is commutative, then \( R_0 = \text{End}(E) \) if and only if 2 remains prime in \( \text{End}(E) \).

When \( \text{End}(E) \) is an order in an imaginary quadratic extension of \( \mathbb{Q} \) so that \( \text{End}(E) = \mathbb{Z} + \mathbb{Z}(D + \sqrt{D}/2) \subset \mathbb{C} \), where \( D < 0 \), this happens if and only if \( D \equiv 5 \mod(8) \). Otherwise, \( R_0 = \{r \in \text{End}(E)| r \equiv \lambda \text{id}(\mod 2 \text{End}(E)), \lambda \in \mathbb{Z}/2\mathbb{Z} \} \).

In the case when \( p \in E_2 \) is nonzero, we can choose a basis \( \{e_1, e_2\} \) in \( E_2 \) with \( e_1 = p \) and use the corresponding identification of \( \text{End}(E_2) \) with \( \mathbb{M}(2,\mathbb{F}_2) \). Then the above proposition implies that a \( \Sigma_{R,e} \)-structure with \( L(1) = \mathcal{O}(e) \) exists if and only if the image of \( R \) in \( \mathbb{M}(2,\mathbb{F}_2) \) is a subalgebra \( R \subset \mathbb{M}(2,\mathbb{F}_2) \) such that for every \( T \in R \setminus \{0, 1\} \) one has either \( e_1 \notin \text{im}(T) \) or \( e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). One can easily show that besides \( \mathbb{F}_2 \subset \mathbb{M}(2,\mathbb{F}_2) \) there are only two more subalgebras in \( \mathbb{M}(2,\mathbb{F}_2) \) having this property (both isomorphic to \( \mathbb{F}_2 \times \mathbb{F}_2 \)): one is generated over \( \mathbb{F}_2 \) by the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and the other is generated by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). In particular, \( R \) has no nilpotents. This proves the “only if” part of the following theorem.

**Theorem 1.4.2.** A \( \Sigma_{R,e} \)-structure for \( \phi_0 \) exists if and only if the image of \( R \) in \( \text{End}(E_2) \) is a ring without nilpotents.

**Proof.** Let \( \bar{R} \subset \text{End}(E_2) \) be a ring without nilpotents. Then either \( \bar{R} \) is a field, or it contains a nontrivial idempotent. In the former case, \( \bar{R} \) is contained in \( \mathbb{F}_4 \subset \text{End}(E_2) \). Otherwise, we can choose a base in \( E_2 \) in such a way that \( \bar{R} \) contains \( E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), and hence it contains the subalgebra \( D = \mathbb{M}(2,\mathbb{F}_2) \) of diagonal matrices. Since \( \bar{R} \) is without nilpotents, this implies that \( \bar{R} = D \).

If \( \bar{R} \) is a field, then \( L(1) = \mathcal{O}(e) \) defines a \( \Sigma_{R,e} \)-structure as we have seen above. Otherwise, for some basis \( \{e_1, e_2\} \) of \( E_2 \), the subalgebra \( \bar{R} \) coincides with \( D = \mathbb{M}(2,\mathbb{F}_2) \approx \text{End}(E_2) \). Now the conditions of Proposition 1.4.1 are satisfied for \( p = e_1 + e_2 \); hence, \( L(1) = \mathcal{O}(p) \) defines a \( \Sigma_{R,e} \)-structure in this case.

**1.5. Existence of** \( \Sigma_{R,e} \)-**structure.** Consider first the case when \( R \) is a commutative integral domain finite over \( \mathbb{Z} \) and \( \epsilon = \text{id} \), that is, \( R = R^+ \) (the case of real multiplication). Then the homomorphism \( \phi : A \to \bar{A} \) above should be just \( R \)-linear. We say that \( R \) is unramified at 2 if \( R/2R \) has no nilpotents.
Proposition 1.5.1. Let $A$ be an abelian variety with multiplication by $R$, and let $M$ be a symmetric line bundle on $A$ such that $\phi = \phi_M$ is $R$-linear. Assume that $R$ is unramified at 2. Then there exists a unique $\Sigma_{R,\id}$-structure for $\phi$ with $L(1) = M$.

Proof. Since $R/2R$ is a product of fields, the Frobenius homomorphism $F : R/2R \to R/2R : x \mapsto x^2$ is bijective. Hence, any element $r \in R$ can be represented in the form $r = a^2 + 2b$ with $a, b \in R$, and if $a^2 + 2b_1 = a_2^2 + 2b_2$, then $a_2 - a_1 \in 2R$. Now we define the $\Sigma_{R,\id}$-structure

$$L(r) := a^* M \otimes ([b], \phi)^* P,$$

where $r = a^2 + 2b$. It is easy to check that $L(r)$ is well defined and satisfies the required properties. The uniqueness follows from (1.3.2) and (1.3.3). \qed

Returning to the general case, let us describe an obstruction to the existence of a $\Sigma_{R,e}$-structure for a given $\phi$. For this we need to assume that the ground field is algebraically closed of characteristic $\neq 2$. Consider the group

$$\tilde{K}(\phi) = \{(L, r_0) \mid L \in \text{Pic}^+(A), r_0 \in R^+, \phi_L = \phi \circ [r_0]_A\}$$

with the group law $(L, r_0)(L', r_0') = (L \otimes L', r_0 + r_0')$. We have an exact sequence of abelian groups

$$0 \to \text{Pic}_2(A) \to \tilde{K}(\phi) \xrightarrow{\pi} R^+ \to 0, \quad (1.5.1)$$

where the first embedding is $\eta \mapsto (\eta, 0)$, $\eta \in \text{Pic}_2(A)$, and $\pi$ is the projection $(L, r_0) \mapsto r_0$. Moreover, we have a canonical splitting $\sigma$ of the pull-back of this extension by the homomorphism $\text{tr} : R \to R^+ : r \mapsto e(r) + r$.

$$\sigma(r) = (([r], \phi)^* P, e(r) + r).$$

Note that if a $\Sigma_{R,e}$-structure for $\phi$ exists, then for $r \in R^-$ we get that the line bundle $([r], \phi)^* P \simeq L(0)$ is trivial. Thus, the first obstacle to existence of such a structure is given by a homomorphism

$$\delta(\phi) : R^- \to \text{Pic}_2(A) : r \mapsto ([r], \phi)^* P.$$

The inclusion $([r], \phi)^* P \in \text{Pic}_2(A)$ follows from the isomorphism

$$([r], \phi)^* P \simeq (\text{id}, [r])^* P \simeq ([e(r)], \phi)^* P.$$

This isomorphism implies also that $\delta(\phi)$ factors through a homomorphism $\delta(\phi) : R^-/\text{tr}^-(R) \to \text{Pic}_2(A)$, where $\text{tr}^-(r) = r - e(r)$. Notice that $\delta$ can be con-
considered as a morphism of right $R/2R$-modules, where the action of $R/2R$ on $\text{Pic}_2(A)$ is given by $r(\eta) = r^*(\eta)$, while its action on $R^+ / \text{tr}^-(R)$ is given by $r(r') = \overline{r(r')} \mod (\text{tr}^-(R))$, where $r \in R$, $r' \in R^+ / \text{tr}^-(R)$.

Assume that $\delta(\phi) = 0$. Then $\sigma$ descends to a splitting $\tilde{\sigma} : \text{tr}(R) \to K(\phi)$ of $\pi$ over the subgroup $\text{tr}(R) \subset R^+$. Hence, we can define the reduced group $K(\phi) = \tilde{K}(\phi) / \sigma(\text{tr}(R))$, which is an extension of $R / \text{tr}(R)$ by $\text{Pic}_2(A)$. It is easy to see that the group $K(\phi)$ has a natural structure of right $R/2R$-module induced by the action $r(L, r') = (r^* L, \varepsilon(r)r')$, so we can consider the exact sequence

$$0 \to \text{Pic}_2(A) \to K(\phi) \to R^+ / \text{tr}(R) \to 0 \quad (1.5.2)$$

as an extension of $R/2R$-modules, where $R^+ / \text{tr}(R)$ is equipped with the following (right) $R/2R$-module structure: $r(ro) = \overline{r(r')} \mod (\text{tr}(R))$ for $r \in R$, $r_0 \in R^+$.

**Proposition 1.5.2.** Assume that the ground field is algebraically closed of characteristic $\neq 2$. Then a $\Sigma_{R, \varepsilon}$-structure for $\phi$ exists if and only if $\delta(\phi) = 0$ and the class $e(\phi) \in \text{Ext}_{R/2R}((R^+ / \text{tr}(R), \text{Pic}_2(A))$ of the extension (1.5.2) is trivial.

**Proof.** We have seen that the condition $\delta(\phi) = 0$ is necessary for existence of a $\Sigma_{R, \varepsilon}$-structure for $\phi$. Also, such a structure gives a splitting $r_0 \mapsto (L(r_0), r_0)$ of the extension (1.5.1), which induces an $R/2R$-linear splitting of (1.5.2). Since all the steps in the argument are invertible, the "if" part follows easily. \[2\]

**Remark.** Notice that in the case of the standard polarization $\phi_0$ of an elliptic curve the homomorphism $\delta(\phi_0)$ can be nontrivial. Indeed, the triviality of this homomorphism is equivalent to the triviality of the line bundle $\mathcal{O}(E_{r-1} - E_r - e)$ for any $r \in R^-$ where we denote $E_r = r^{-1}(e)$. In the case of characteristic zero, this is equivalent to the following identity for the group law on $E$:

$$\sum_{(r-1)x=0} x = \sum_{rx=0} x$$

for any $r \in R^-$. One can see easily that this can happen only when both sides are zero. In particular, if $\ker(r|_{E_0})$ has order 2, but $\ker((r - 1)|_{E_0}) = 0$, then $\delta(\phi_0) \neq 0$. For example, this is so when $R$ contains $r = \sqrt{-2}$, which acts nontrivially on $E_2$.

Let $R$ be an order in a finite-dimensional division algebra $D$ over $\mathbb{Q}$, and let $\varepsilon$ be an involution of $R$, such that the corresponding involution of $D$ is positive, that is, $\text{Tr}_{D/\mathbb{Q}}(\varepsilon(x)x) > 0$ for any $x \in D^*$. Let $K$ be the center of $D$, so that $\mathfrak{o} = R \cap K$ is an order in $K$. Recall (see, e.g., [5]) that if $\varepsilon|_{\mathfrak{o}}$ is trivial, then either $D = K$ or $D$ is a quaternion algebra over $K$, which is either totally indefinite (unramified at every infinite place) or totally definite.

**Theorem 1.5.3.** Assume that $\mathfrak{o}$ is unramified at every $\varepsilon$-stable prime ideal $p$ of $\mathfrak{o}$ above 2 and that $R/pR$ is semisimple. If $\varepsilon|_{\mathfrak{o}}$ is trivial, then assume, in addition,
that either \(D = K\) or that \(D\) is an indefinite quaternion algebra over \(K\) and for every prime \(p \subset \mathfrak{o}\) over 2, the completion \(R_p\) is isomorphic to the matrix algebra \(M_2(\mathfrak{o}_p)\), where \(\mathfrak{o}_p\) is the completion of \(\mathfrak{o}\) at \(p\). Let \(A\) be an abelian variety over an algebraically closed field \(k\) such that \(\text{char}(k) \neq 2\). Then for any symmetric homomorphism \(\phi : A \to A\), such that \(\phi \circ [e(r)]_A = [\hat{r}]_A \circ \phi\) for any \(r \in R\), there exists a \(\Sigma_{R,x}\)-structure for \(\phi\).

**Remark.** Let \(\mathfrak{o}\) be the ring of integers in \(K\) and \(R\) be the maximal \(\mathfrak{o}\)-order in \(D\). Then the conditions of the above theorem are that \(K/\mathbb{Q}\) and \(D/K\) are unramified at every \(\varepsilon\)-stable place of \(K\) above 2 and \(D\) is not a definite quaternion algebra over \(K\) when \(\varepsilon|\mathfrak{o}\) is trivial (is not of Type III in the classification list of [5, IV, 21, Thm. 2]).

We need two lemmas for the proof.

**Lemma 1.5.4.** Let \(\mathbb{F}_{2^l}\) be a finite field with \(2^l\) elements and let \(M = M_n(\mathbb{F}_{2^l})\) be the matrix algebra over \(\mathbb{F}_{2^l}\). Let \(\sigma\) be an involution of \(M\) such that \(\sigma|_{\mathbb{F}_{2^l}}\) is non-trivial. Then for every element \(m_0 \in M\) stable under \(\sigma\), there exists \(m \in M\) such that \(\sigma_0(m_0) + m = m\).

**Proof.** Since \(\sigma^2 = \text{id}\), we should have necessarily \(l = 2d\) and \(\sigma|_{\mathbb{F}_{2^l}}(x) = x^{2^d}\), so for \(m_0 \in \mathbb{F}_{2^l} \subset M\) the assertion follows. Let \(\sigma_0\) be the following involution of \(M\):

\[
\sigma_0((a_{ij})) = (\sigma|_{\mathbb{F}_{2^l}}(a_{ij})).
\]

Then \(\sigma \circ \sigma_0\) is an automorphism of \(M\) that should be inner, and hence, we get \(\sigma(x) = u\sigma_0(x)u^{-1}\) for some \(u \in M^* = \text{GL}_n(\mathbb{F}_{2^l})\) such that \(\sigma_0(u) = \lambda u\) for \(\lambda \in \mathbb{F}_{2^l}^\times\). It follows that \(\lambda^{2^d} = \lambda^{-1}\); that is, \(\lambda = \mu^{2^{d-1}}\) for some \(\mu \in \mathbb{F}_{2^l}^\times\). Thus, changing \(u\) by \(\mu^{-1}u\) we may assume that \(\sigma_0(u) = u\). Note that for \(\sigma_0\) the assertion follows from the case \(m_0 \in \mathbb{F}_{2^l}\) considered above. Now if \(\sigma(m_0) = u\sigma_0(m_0)u^{-1} = m_0\), then \(\sigma_0(m_0)u = m_0u\); therefore, \(m_0u = \sigma_0(m) + m\) for some \(m \in M\), and hence, \(m_0 = \sigma(mu^{-1}) + mu^{-1}\).

**Lemma 1.5.5.** Let \(B\) be a discrete valuation ring. Then any automorphism of the matrix algebra \(M_n(B)\) is inner.

**Proof.** Let \(L\) be the field of fraction for \(B\). Then any automorphism of \(M_n(L)\) is inner; hence, any automorphism of \(M_n(B)\) has form \(\alpha(a) = uau^{-1}\), where \(u \in \text{GL}_n(L)\) is such that \(uM_n(B)u^{-1} = M_n(B)\). Considering the standard left action of \(M_n(L)\) on \(L^n\), we derive the inclusion \(a(uB^n) \subset uB^n\) for any \(a \in M_n(B)\). Let \(\pi \subset B\) be a uniformizing element. Changing \(u\) by a scalar, we may assume that \(uB^n \subset B^n\), but \(uB^n \not\subset \pi B^n\). Then the image of \(uB^n\) in \((B/\pi B)^n\) is invariant under the standard action of \(M_n(B/\pi B)\), which implies that \(uB^n = B^n\), that is, \(u \in \text{GL}_n(B)\).

**Proof of Theorem 1.5.3.** The first step is to show that under the assumptions of the theorem \(R^- = \text{tr}^- (R)\), so that \(\delta'(\phi) = 0\). Since \(2R^- \subset \text{tr}^- (R)\), it is sufficient
to check the inclusion $R^-/2R^- \subset \text{tr}(R)/2R^+$ of subgroups in $R/2R$. Let $(2) = \bigcap_i q_i$ be the primary decomposition of 2 in $\mathfrak{o}$, where $q_i$ are $p_i$-primary ideals and $p_i$ are different prime ideals of $\mathfrak{o}$. Then $R/2R$ contains $\mathfrak{o}/2\mathfrak{o} = \prod_i \mathfrak{o}/q_i$ as a central subalgebra, and there is a decomposition $R/2R \simeq \prod_i M_i$, where $M_i = R/q_i R$. Note that $\varepsilon$ permutes $p_i$, so that $\varepsilon(p_i) = p_{\varepsilon(i)}$; hence,

$$(2) = \bigcap_i \varepsilon(q_i) = \bigcap_i q_{\varepsilon(i)} = \bigcap_i (\varepsilon(q_i) \cap q_{\varepsilon(i)}).$$

Changing $q_i$ by $q_i \cap \varepsilon(q_{\varepsilon(i)})$, we may assume that $\varepsilon(q_i) = q_{\varepsilon(i)}$. Then the induced involution of $R/2R$ maps $M_i$ to $M_{\varepsilon(i)}$. Also, if $\varepsilon(i) = i$, then $q_i = p_i$, and $M_i = R/p_i R$ is semisimple. Let $r \in R^-; \text{ then the image of } r \text{ in } R/2R$ decomposes as follows: $\bar{r} = \sum r_i$, where $r_i \in M_i$, $r_{\varepsilon(i)} = \varepsilon(r_i)$. To prove that $\bar{r} \in \text{tr}(R)/2R^+$, it is sufficient to check that $r_i \in \text{tr}(R)/2R^+$ for every $i$ such that $\varepsilon(i) = i$.

Assume first that $\varepsilon|_\mathfrak{o}$ is nontrivial. Since $\mathfrak{o}$ is unramified at every $\varepsilon$-stable place above 2, the induced involution of $\mathfrak{o}/p_i$ for $\varepsilon(i) = i$ is nontrivial. For such $i$, the $\mathfrak{o}/p_i$-algebra $M_i$ is a product of matrix algebras over field extensions of $\mathfrak{o}/p_i$, and we are done by Lemma 1.5.4.

Now let $\varepsilon|_\mathfrak{o}$ be trivial. In the case $D = K$, we have $R^- = 0$, so we may assume that $D$ is an indefinite quaternion algebra over $K$. Then $M_i = R/p_i R \simeq \tilde{R}_i/p_i \tilde{R}_i$ for every $i$, where $\tilde{R}_i \simeq M_2(\tilde{\mathfrak{o}}_i)$ is the $p_i$-adic completion of $\mathfrak{R}_i$. By Lemma 1.5.5 the induced involution $\varepsilon: \tilde{R}_i \to \tilde{R}_i$ has form $\varepsilon(x) = u x^t u^{-1}$ for some $u \in GL_2(\tilde{\mathfrak{o}}_i)$, where $x^t$ denotes the tranposed matrix to $x$ and $u^t = \pm u$. We claim that the case $u^t = -u$ is impossible. Indeed, let $x \mapsto x^* = \text{Tr}_{D/K}(x) - x$ be the canonical involution of $D$, where $\text{Tr}_{D/K} : D \to K$ is the reduced trace. Then for the involution $\varepsilon$ on $D$, we have $\varepsilon(x) = a x^* a^{-1}$ for some $a \in D^*$ such that $a^* = -a$ (see [5]). It follows that for the induced involution of the $p_i$-adic completion $\tilde{D}_i \simeq M_2(\tilde{\mathfrak{o}}_i)$, we have $\varepsilon(x) = a x^* a^{-1} = u x^t u^{-1}$. Note that $x^* = s x^t s^{-1}$, where $s = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$; hence, $u$ is proportional to $as$, and the condition $a^* = -a$ rewritten as $(as)^t = as$ implies that $u^t = u$. Therefore, if $\varepsilon(x) = -x$ for some $x \in \tilde{R}_i$, then $x = y - \varepsilon(y)$ for $y \in \tilde{R}_i$, which implies the required inclusion $r_i \in \text{tr}(R)/2R^+$.

By Proposition 1.5.2 it remains to show that the extension of $R/2R$-modules (1.5.2) splits. When $\varepsilon|_\mathfrak{o}$ is trivial, this is a consequence of the semisimplicity of $R/2R$. Otherwise, the argument above shows that $R^+/\text{tr}(R) = 0$.

If $\varepsilon = \text{id}$ and $R = \mathfrak{o}$, we can improve the above theorem as follows.

**Theorem 1.5.6.** Let $\mathfrak{o}$ be an order in the number field. Assume that the localization of $\mathfrak{o}$ at every prime ideal above 2 is normal. Let $A$ be an abelian variety over an algebraically closed field $k$ such that $\text{char}(k) \neq 2$. Then for any $\mathfrak{o}$-linear polarization $\phi : A \to \hat{A}$ there exists a $\Sigma_{\varepsilon, \text{id}}$-structure for $\phi$.

**Proof.** Since $R^- = 0$ in this case, it is sufficient to show that $\text{Ext}^1_{R/2R}(M, \hat{A}) = 0$ for any finite $R/2R$-module $M$. Since $(2) = \bigcap_i p_i^t$ for different prime ideals $p_i \subset \mathfrak{o}$, it is sufficient to prove that $\text{Ext}^1_{R_i}(k_i, \hat{A}_{p_i^t}) = 0$, where
Now we use the following general fact: If $B$ is a discrete valuation ring with a uniformizing element $\pi$ and $N$ is a $B/(\pi^n)$-module such that the natural map $N \to N_{\pi^{-1}} = \{x \in N \mid \pi^{-1}x = 0\}$ induced by the action of $\pi$ is surjective, then $\text{Ext}^1_{B/(\pi^n)}(B/(\pi), N) = 0$. Indeed, this follows easily from the resolution $0 \to B/(\pi^{n-1}) \xrightarrow{\pi} B/(\pi^n) \to B/(\pi) \to 0$ for $B/(\pi)$. Thus, it is sufficient to check the surjectivity of the homomorphism $\hat{A}_{p_i^l} \to \hat{A}_{p_i^{l-1}}$ induced by the action of the local uniformizer $\pi \in p_i$. Note that $(\pi) = p_i q$ for some nonzero ideal $q$ prime to $p_i$. Hence, we have a decomposition $\hat{A}_{p_i^l} \simeq \hat{A}_{p_i^{l'}} \times \hat{A}_{q_i^{n-l}}$. Since $[\pi] : A \to \hat{A}$ is an isogeny, the homomorphism $\hat{A}_{p_i^l} \to \hat{A}_{p_i^{l-1}}$ is surjective, which implies the surjectivity of the induced homomorphism $\hat{A}_{p_i^l} \to \hat{A}_{p_i^{l-1}}$.

2. Theta functions

2.1. Canonical theta function. Our notation below is close to [1]. The only substantial difference is that we write the canonical theta series in slightly more invariant form.

Let $V$ be a complex vector space with a positive-definite hermitian form $H$, and let $L \subset V$ be a $\mathbb{Z}$-lattice such that the restriction of $E = \text{Im} H$ to $L$ takes integer values. Let $\chi : L \to \mathbb{C}_1^* = \{z \in \mathbb{C} : |z| = 1\}$ be a map such that

$$\chi(l_1 + l_2) = \chi(l_1)\chi(l_2)\exp(\pi iE(l_1, l_2)).$$

(2.1.1)

A canonical theta function for $(H, \chi)$ is a holomorphic function $f$ on $V$ such that

$$f(v + l) = \chi(l)\exp(\pi H(v, l) + \frac{\pi}{2} H(l, l)) f(v).$$

We denote the space of such functions by $T(H, L, \chi)$. One can interpret this condition as invariance of $f$ under the action of some group. Namely, let $\text{Heis}(V)$ be the Heisenberg group corresponding to $(V, E)$. Recall that as a set $\text{Heis}(V)$ is defined by the formula

$$(t, v) \cdot (t', v') = (tt' \exp(\pi iE(v, v')), v + v'),$$

where $t, t', v, v' \in V$. In particular, $\text{Heis}(V)$ is a central extension of $V$ by $\mathbb{C}_1^*$. There is a representation of $\text{Heis}(V)$ on the space of holomorphic functions on $V$ given by the formula

$$U(t, v)f(x) = t^{-1}\exp\left(-\pi H(x, v) - \frac{\pi}{2} H(v, v)\right)f(x + v),$$

where $U(t, v)$ is an operator corresponding to $(t, v) \in \text{Heis}(V)$. It is easy to see from (2.1.1) that the map $l \mapsto (\chi(l), l)$ defines a homomorphism $\sigma_\chi : L \to \text{Heis}(V)$. Now the definition of a canonical theta function can be rephrased as the condition that $f$ is invariant under the action of $\sigma_\chi(L)$. In particular, the normalizer
$N_\chi \subset \text{Heis}(V)$ of the subgroup $\sigma_\chi(L) \subset \text{Heis}(V)$ acts on the space $T(H,L,\chi)$ of canonical theta functions for $(H,\chi)$. It is easy to see that $N_\chi$ consists of elements $(t,v) \in \text{Heis}(V)$ with $v \in L^\perp$, where $L^\perp = \{v \in V : E(v,L) \subset \mathbb{Z}\}$. Furthermore, it is known that $T(H,L,\chi)$ is an irreducible representation of the group $G(H,L,\chi) = N_\chi/\sigma_\chi(L)$ of dimension $\sqrt{[L^\perp : L]}$. Recall also that $T(H,L,\chi)$ is identified with the space of global sections of the line bundle $L(H,\chi)$ on the complex abelian variety $V/L$ (see, e.g., [5]), and the action of $G(H,L,\chi)$ on it can be defined in purely algebraic terms.

An example of a map $\chi$ satisfying (2.1.1) is obtained when we have a decomposition $L = L_1 \oplus L_2$, where $L_i$ are isotropic with respect to $E$. (Further, we refer to such decomposition as isotropic decomposition of $L$.) Namely, there is a canonical map $\chi_0 = \chi_0(L_1,L_2) : L \to \{\pm 1\}$ satisfying (2.1.1), which is given by the formula

$$\chi_0(l) = \exp(\pi i E(l_1,l_2)),$$

where $l = l_1 + l_2$, $l_i \in L_i$. Any two maps $\chi$ and $\chi'$ satisfying (2.1.1) are related by the formula

$$\chi'(l) = \chi(l) \exp(2\pi i E(c,l))$$

for some $c \in V$, which is uniquely determined modulo $L^\perp$. It is easy to see that the corresponding homomorphisms $\sigma_\chi'$ and $\sigma_\chi$ are related as follows:

$$\sigma_\chi'(l) = (1,c)\sigma_\chi(l)(1,c)^{-1}.$$  

Therefore, we can define an isomorphism of the corresponding finite Heisenberg groups

$$\alpha_c : G(H,L,\chi) \to G(H,L,\chi') : g \mapsto (1,c)g(1,c)^{-1}.$$  

Now the operator $U(1,c)$ restricts to an isomorphism between $T(H,L,\chi)$ and $T(H,L,\chi')$ compatible with the actions of $G(H,L,\chi)$ and $G(H,L,\chi')$ via $\alpha_c$.

Now assume that we have data $(H,L,\chi)$ as above and assume that $U \subset V$ is a maximal $E$-isotropic $\mathbb{R}$-subspace such that $U$ is generated by $U \cap L$ over $\mathbb{R}$, and $\chi|_{U \cap L} \equiv 1$. It is easy to see that $U$ generates $V$ as a $\mathbb{C}$-space and since $H|_{U \times U}$ is a symmetric form, it extends to a $\mathbb{C}$-bilinear symmetric form $S : V \times V \to \mathbb{C}$. Now we set

$$\theta_{H,L,U}^\chi(x) = \exp \left( \frac{\pi}{2} S(x,x) \sum_{l \in L \cap U \cap L} \chi(l) \exp(\pi i H - S)(x,l) - \frac{\pi}{2} (H - S)(l,l) \right).$$  

(2.1.6)
One can easily check that $\theta_{H,L,U}^\chi \in T(H,L,\chi)$. Furthermore, notice that $\bar{L} = L + U \cap L_\perp$ is also a lattice in $V$ such that the restriction of $E$ to $\bar{L}$ is integer-valued. The map $\chi$ has a unique extension to a map $\bar{\chi} : \bar{L} \to \mathbb{C}_1^*$ satisfying (2.1.1), such that $\bar{\chi}|_{U \cap L_\perp} \equiv 1$. Then one has

$$\theta_{H,L,U}^\chi = \theta_{\bar{L},L,U}^{\bar{\chi}}.$$  

In particular, $\theta_{H,L,U}^\chi$ is an element of $T(H,\bar{L},\bar{\chi}) \subset T(H,L,\chi)$. In other words, $\theta_{H,L,U}^\chi \in T(H,L,\chi)$, and $\theta_{H,L,U}^\chi$ is invariant under the action of $(1, U \cap L_\perp) \subset G(H,L,\chi)$.

**Lemma 2.1.1.** For any $c \in U$ one has

$$\theta_{H,L,U}^\chi = U_{(1,c)} \theta_{H,L,U}^\chi,$$

where $\chi'$ and $\chi$ are related by (2.1.3).

The proof is straightforward and is left to the reader.

The following simple statement is sometimes referred to as the “Isogeny theorem.”

**Lemma 2.1.2.** Let $H, L, \chi, U$ be as above and let $L' \subset L$ be a sublattice. Then

$$\theta_{H,L,U}^\chi = \sum_{l \in L/(L' + U \cap L)} \chi(l)^{-1} U_{(1,l)} \theta_{H,L',U}^\chi.$$

We also need the following lemma (in which $V$ can be replaced by any real symplectic vector space).

**Lemma 2.1.3.** If $L \subset V$ and $U$ are as above, then the lattice $\bar{L} = L + U \cap L_\perp$ is self-dual.

**Proof.** It is sufficient to prove that if $L$ and $U$ are as above and $U \cap L_\perp = U \cap L$, then $L$ is self-dual. (To prove the statement of the lemma, apply this to $\bar{L}$.) We use the induction in the dimension of $V$. Choose a nonzero element $x \in U \cap L$. Then there exists $N \in \mathbb{Z}$ such that $E(x,L) = N\mathbb{Z} \subset \mathbb{Z}$. In particular, $x/N \in U \cap L_\perp = U \cap L$. Replacing $x$ by $x/N$ we can assume that $N = 1$, so that there exists an element $y \in L$ such that $E(x,y) = 1$. Consider the $E$-orthogonal decomposition $V = (\mathbb{R}x \oplus \mathbb{R}y) \oplus V_0$. Then $L = (\mathbb{Z}x \oplus \mathbb{Z}y) \oplus V_0 \cap L$, $L_\perp = (\mathbb{Z}x \oplus \mathbb{Z}y) \oplus V_0 \cap L_\perp$ and $U = \mathbb{R}x \oplus V_0 \cap U$. Hence, we can apply the induction assumption to $V_0 \cap L$ and $V_0 \cap U$. □

**Remarks.**

1. When one has an isotropic decomposition $L = L_1 \oplus L_2$ such that $U \cap L_1 = L_2$ and $\chi = \chi_0(L_1,L_2)$, the function $\theta_{H,L,U}^\chi$ we defined coincides with the function $\vartheta^0$ defined in [1, Ch. 3, 2.3].

2. If $L = L_\perp$, then for given $H$ and $\chi$, an isotropic subspace $U$ as above exists if and only if the line bundle $\mathcal{L}(H,\chi)$ on $V/L$ is even (see [6]).
2.2. Classical theta functions and the functional equation. Let \( Z \) be an element of the Siegel upper half-plane \( \mathcal{H} \); that is, let \( Z \) be a \( g \times g \) matrix, such that \( Z^t = Z \) and Im \( Z > 0 \). Then it defines an abelian variety with principal polarization in the standard way. First, \( L(Z) = \mathbb{Z}Z \oplus \mathbb{Z}^g \) is a lattice in \( \mathbb{C}^g \), and the hermitian form \( H_Z \) on \( \mathbb{C}^g \) is defined by the matrix \((\text{Im} Z)^{-1}\) in the standard basis. Then one has an isotropic decomposition \( L(Z) = L(Z)_1 \oplus L(Z)_2 \), where \( L(Z)_1 = \mathbb{Z}Z \), \( L(Z)_2 = \mathbb{Z}^g \); hence the corresponding map \( \chi_0 : L(Z) \to \{ \pm 1 \} \), satisfying (2.1.1). One also has the corresponding decomposition \( \mathbb{C}^g = \mathbb{R}^g \oplus \mathbb{R}^g \) into summands that are lagrangian with respect to the real symplectic form \( E_Z = \text{Im} H_Z \). For \( v \in \mathbb{C}^g \) we use the notation \( v = Zv_1 + v_2 \), where \( v_1, v_2 \in \mathbb{R}^g \).

Now one can compute that for any \( c \in \mathbb{C}^g \), one has

\[
U(1,c) \theta_{H_Z,L(Z),\mathbb{R}^g} = \exp \left( \frac{\pi}{2} S(\cdot, \cdot) - \pi i (c_1)^t \cdot c_2 \right) \theta \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right] (\cdot, Z),
\]

where \( S(v, w) = v^t (\text{Im} Z)^{-1} w \), \( \theta \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right] (\cdot, Z) \) is the classical theta function with characteristics

\[
\theta \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right] (v, Z) = \sum_{l \in \mathbb{Z}^g} \exp(\pi i (l + c_1)^t Z (l + c_1) + 2\pi i (v + c_2)(l + c_1))
\]

for \( v \in \mathbb{C}^g \).

We are going to use this comparison and rewrite the classical functional equation in terms of canonical theta functions. Namely, assume that we have a complex vector space \( V \), a lattice \( L \subset V \), and a positive hermitian form \( H \) on \( V \) such that the restriction of \( E = \text{Im} H \) to \( L \) takes integer values and \( L^\perp = L \). Then to every pair \( (\chi, U) \), where \( \chi : L \to \mathbb{C}^* \) is a map satisfying (2.1.1) and \( U \subset V \) is an \( E \)-lagrangian subspace generated by \( U \cap L \) such that \( \chi|_U \equiv 1 \), we associated the canonical theta function \( \theta_{H,L,U}^\chi \). Now if we consider another such pair \( (\chi', U') \), then we get the canonical theta function \( \theta_{H,L,U'}^{\chi'} \) for \( (H, \chi') \). We can choose \( c \in V \) (uniquely up to adding an element of \( L \)) such that

\[
\chi'(l) = \chi(l) \exp(2\pi i E(c, l)) \quad (2.2.1)
\]

for \( l \in L. \) Then \( U_{1,c} \) gives an isomorphism of \( T(H, L, \chi) \) with \( T(H, L, \chi') \). Since \( T(H, L, \chi') \) in this case is 1-dimensional, we should have an identity

\[
\theta_{H,L,U'}^{\chi'}(v) = q \cdot U_{1,c} \theta_{H,L,U}^{\chi}(v), \quad (2.2.2)
\]

where \( q \in \mathbb{C}^* \) is a constant depending on \( H, \chi, c, U, \) and \( U' \).

For every pair \( M_1, M_2 \) of free \( \mathbb{Z} \)-modules of rank \( g = \dim V \) in \( V \) such that \( M_i \) generates \( V \) over \( \mathbb{C} \), we define \( \det_{M_i}(M_2) \in \mathbb{C}^*/\{ \pm 1 \} \) as follows: choose arbitrary bases of \( M_i \) and write the transition matrix from the basis in \( M_1 \) to that in \( M_2 \),
then take its determinant. Up to sign, this number does not depend on a choice of bases in $M_i$.

**Theorem 2.2.1.** Let $(\chi, U)$ and $(\chi', U')$ be as above. Assume also that $\chi^2 \equiv \chi'^2 \equiv 1$. Then for any $c \in (1/2)L$, such that (2.2.1) holds, one has

$$
\theta_{H, L, U}(v) = \zeta \cdot \det_{U \cap L}(U' \cap L)^{1/2} U_{(1, c)} \theta_{H, L, U}(v),
$$

(2.2.3)

where $\zeta^8 = 1$.

**Proof.** First let us assume that $\chi = \chi_0(L_1, L_2)$, $U = RL_2$ for some isotropic decomposition $L = L_1 \oplus L_2$, and similarly the pair $(\chi', U')$ arises from some isotropic decomposition $L = L'_1 \oplus L'_2$. Then we can find an automorphism $T : L \to L$, which preserves $E|_{L \times L}$, such that $L'_i = T(L_i)$, $i = 1, 2$. Choosing bases in $L_1$ and $L_2$ in such a way that the matrix of $E|_{L \times L}$ with respect to them is standard, and identifying $V$ with $0$, using the base in $L_2$, we may assume that $V = C^g$, $H = H_2$, $L_1 = ZZ^g$, and $L_2 = Z^g$ for some $Z \in S_g$. Let $(e_1, \ldots, e_g)$ be the standard basis in $Z^g$; then $(Ze_1, \ldots, Ze_g, e_1, \ldots, e_g)$ is the basis of $L$ in which $E$ has the standard form. With respect to this base, $T$ is given by a symplectic matrix $[T] \in Sp_{2g}(Z)$. Let $[T] = ([A] [C])$ be the block form of $[T]$, where $A, B, C, D \in M(g, Z)$. Then $L'_1 = (ZA + B)(Z^g) \subset C^g$ and $L'_2 = (ZC + D)(Z^g) \subset C^g$. Thus, $(ZC + D)^{-1}(L'_2) = Z^g$ and $(ZC + D)^{-1}(L'_i) = Z^g$, where $Z' = (ZC + D)^{-1}(ZA + B) \in S_g$. It follows that

$$
\theta_{H, L_1, L_2}^{\chi_0(L_1, L_2)}((ZC + D)v) = \theta_{H, L_1, L_2}^{\chi_0(L_1, L_2)}((ZC + D)v),
$$

so that (2.2.2) with $v = 0$ assumes the form

$$
\theta \left[ \begin{array}{c}
0 \\
0
\end{array} \right](0, Z') = q \cdot \exp(-\pi i(c_1) \cdot c_2) \cdot \theta \left[ \begin{array}{c}
c_1 \\
c_2
\end{array} \right](0, Z).
$$

(2.2.4)

Comparing this with the classical functional equation and using the fact that $c \in (1/2)L$, we conclude that

$$
q = \zeta \cdot \det(ZC + D)^{1/2},
$$

where $\zeta^8 = 1$. Now by definition $\det(ZC + D)$ represents $\det_{L_2}(L_2') \in \mathbb{C}^*/\{\pm 1\}$. Hence, we can rewrite (2.2.2) in the form

$$
\theta_{L_1, L_2}^{\chi_0(L_1, L_2)}(v) = \zeta \cdot \det_{L_2}(L_2')^{1/2} \cdot U_{(1, c)} \theta_{L_1, L_2}^{\chi_0(L_1, L_2)}(v),
$$

(2.2.5)

where $\det_{L_2}(L_2')^{1/2}$ is defined up to multiplying by the 4th root of unity and $\zeta$ is an 8th root of unity defined with the same ambiguity.
The general case can be deduced as follows. We can always choose isotropic decompositions $L = L_1 \oplus L_2$ and $L = L_1' \oplus L_2'$ such that $U = \mathbb{R}L_2$ and $U' = \mathbb{R}L_2'$. Then we can find $c_1 \in U \cap (1/2)L$ and $c_2 \in U' \cap (1/2)L$ such that

$$\chi = \chi_0(L_1, L_2) \exp(2\pi i E(c_1, \cdot)),
\chi' = \chi_0(L_1', L_2') \exp(2\pi i E(c_2, \cdot)).$$

Then by Lemma 2.1.1 we have

$$\theta_{H, L, U}^\chi = U_{(1, c_1)} \theta_{H, L, U}^{\chi_0(L_1, L_2)},
\theta_{H, L, U'}^{\chi'} = U_{(1, c_2)} \theta_{H, L, U'}^{\chi_0(L_1', L_2')},$$

and the equation is easily deduced from the case considered above. \hfill \Box

2.3. Theta identity. Let $V, L, H, \chi$ be as in Section 2.1. Assume that $V/L$ has a complex multiplication by a ring $R$ and that $\varepsilon : R \to R$ is an involution such that $H(\varepsilon(r)v, v') = H(v, rv')$.

Let $B = (b_{ij}) \in M(k, R)$ be a matrix such that $B^\varepsilon \cdot B = n \cdot \text{Id}$ for some $n \in \mathbb{Z}_{>0}$. Here $B^\varepsilon = \varepsilon(B)^t$, where $\varepsilon(B)$ is obtained by applying $\varepsilon$ to all elements of $B$. In other words, if we consider the morphism of free, right $R$-modules $B : R^k \to R^k$ and the standard hermitian form $h_1(X, Y) = X^\varepsilon \cdot Y$ on $R^k$ (here we represent elements of $R^k$ as columns), then one has

$$B^{-1}h_1^k = nh_1^k. \quad (2.3.1)$$

Then if we consider $B$ as a complex operator on $V \otimes^k$, one can easily check that

$$B^{-1}H^\otimes^k = nH^\otimes^k. \quad (2.3.2)$$

This implies that we have a map

$$B^* : T(H^\otimes^k, L^\otimes^k, \chi^\otimes^k) \to T(nH^\otimes^k, L^\otimes^k, B^{-1}(\chi^\otimes^k)) : f \mapsto f(B(\cdot)),
$$

where $\chi^\otimes^k(l_1, \ldots, l_k) = \prod_i \chi(l_i)$. Furthermore, this map is compatible with the actions of the corresponding Heisenberg groups on these spaces via the homomorphism

$$G(nH^\otimes^k, L^\otimes^k, (\chi^\otimes^k)^\otimes^k) \to G(H^\otimes^k, L^\otimes^k, \chi^\otimes^k) : (t, v) \mapsto (t, B(v)),$

where $v \in (n^{-1}L^\perp)^\otimes^k$. 
Now assume that $\chi^2 \equiv 1$ and that we have an $E$-lagrangian subspace $U \subset V$ generated by $U \cap L$ such that $\chi|_{U \cap L} = 1$. Let $\tilde{L} = L + U \cap L^\perp$ and let $\tilde{\chi} : \tilde{L} \to \{\pm 1\}$ be the unique extension of $\chi$ to $\tilde{L}$ satisfying (2.1.1) such that $\tilde{\chi}|_{U \cap L^\perp} \equiv 1$. Then, according to Lemma 2.1.3, the lattice $\tilde{L}$ is self-dual with respect to $E$.

**Lemma 2.3.1.** There exists an element $c \in ((1/2n)L^\perp)^{\oplus k}$ such that

$$\chi^{\oplus k}(Bl) = (\chi^n)^{\oplus k}(l) \exp(2\pi inE^{\oplus k}(c, l))$$

for any $l \in L^{\oplus k}$, and

$$\tilde{\chi}^{\oplus k}(Bv) = \exp(2\pi inE^{\oplus k}(c, v))$$

for any $v \in U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k})$.

**Proof.** First choose $c' \in (1/2)B^{-1}(\tilde{L}^{\oplus k})$ such that

$$B^{-1}(\tilde{\chi}^{\oplus k})_{|U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k})} = \exp(2\pi inE^{\oplus k}(c', \cdot)).$$

Now we define a map $\chi' : B^{-1}(\tilde{L}^{\oplus k}) \to \{\pm 1\}$ by the formula

$$B^{-1}(\tilde{\chi}^{\oplus k}) = \chi' \exp(2\pi inE^{\oplus k}(c', \cdot)).$$

Then $\chi'|_{U^{\oplus k} \cap L^{\oplus k}} \equiv 1$, so we can choose an element $c'' \in U^{\oplus k} \cap ((1/2n)L^\perp)^{\oplus k}$ such that

$$\chi'|_{L^{\oplus k}} = (\chi^n)^{\oplus k}(c'', \cdot).$$

It remains to set $c = c' + c''$.

**Theorem 2.3.2.** With the above notation, one has

$$B^*\theta^{\oplus k}_{H^{\oplus k}, L^{\oplus k}, U^{\oplus k}} = \zeta \cdot \det B^{-1/2}n^{\oplus k/2}d^{-1/2} \cdot \sum_v \chi(Bv)U_{(1,v)}U_{(1,c)}\theta^{(\chi^n)^{\oplus k}}_{nH^{\oplus k}, L^{\oplus k}, U^{\oplus k}},$$

(2.3.3)

where $\det B$ is the determinant of $B$ considered as a complex operator on $V^k$, the summation is taken over the finite group $v \in B^{-1}(\tilde{L}^{\oplus k})/(L^{\oplus k} + U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}))$, $d$ is the number of elements in this group, the Heisenberg action on the right-hand side is associated with the hermitian form $nH^{\oplus k}$, and an element $c$ is chosen as in Lemma 2.3.1.

**Proof.** Notice that $B^{-1}(\tilde{L}^{\oplus k})$ is a self-dual lattice with respect to $B^{-1}E^{\oplus k} = nE^{\oplus k}$, and one has

$$B^*\theta^{\oplus k}_{H^{\oplus k}, L^{\oplus k}, U^{\oplus k}} = \theta^{B^{-1}(\tilde{\chi}^{\oplus k})}_{nH^{\oplus k}, B^{-1}(\tilde{L}^{\oplus k}), B^{-1}(U^{\oplus k})},$$
Now we want to apply the functional equation (2.2.3) to the self-dual lattice $B^{-1}(\bar{L} \oplus k)$ and a pair of lagrangian subspaces $B^{-1}(U \oplus k)$ and $U \oplus k$ in $V \oplus k$. Let us define a map $\chi : B^{-1}(\bar{L} \oplus k) \to \{ \pm 1 \}$ by the formula

$$B^{-1}(\bar{\chi} \oplus k) = \chi \exp(2\pi i n E \oplus k(c, \cdot)),$$

where $c$ is chosen as in Lemma 2.3.1. Then $\chi'(v_1 + v_2) = \chi'(v_1)\chi'(v_2) \cdot \exp(2\pi i n E \oplus k(v_1, v_2))$ and $\chi'|_{U \oplus k \cap B^{-1}(\bar{L} \oplus k)} \equiv 1$. Applying (2.2.3) we get

$$\theta_{nH \oplus k, B^{-1}(\bar{L} \oplus k), B^{-1}(U \oplus k)}^{B^{-1}(\bar{\chi} \oplus k)} = \zeta \cdot \det_{U \oplus k \cap B^{-1}(\bar{L} \oplus k)}(B^{-1}(U \oplus k) \cap B^{-1}(\bar{L} \oplus k))^{1/2} \chi'(v_1) U(1, v) U(1, c) \chi'(v_2) U(1, v) U(1, c). \quad (2.3.4)$$

Now we apply Lemma 2.1.2 to the embedding of lattices $L \oplus k \subset B^{-1}(\bar{L} \oplus k)$ and use the fact that $\chi'|_{L \oplus k} = (\chi'^* \oplus k)^*$:

$$\theta'_{nH \oplus k, B^{-1}(\bar{L} \oplus k), U \oplus k}^{B^{-1}(\bar{\chi} \oplus k)} = \sum_{v \in B^{-1}(\bar{L} \oplus k)/(U \oplus k + U \oplus k \cap B^{-1}(\bar{L} \oplus k))} \chi'(v) U(1, v) \theta_{nH \oplus k, L \oplus k, U \oplus k}^{(\chi'^*) \oplus k}. \quad (2.3.5)$$

Combining (2.3.4) and (2.3.5), we obtain

$$B^* \theta_{H \oplus k, L \oplus k, U \oplus k}^{\chi \oplus k} = \zeta \cdot \det_{U \oplus k \cap B^{-1}(\bar{L} \oplus k)}(B^{-1}(U \oplus k) \cap B^{-1}(\bar{L} \oplus k))^{1/2} \sum_{v} \chi'(v) U(1, v) U(1, c) \theta_{nH \oplus k, L \oplus k, U \oplus k}^{(\chi'^*) \oplus k},$$

where the summation is taken over $v \in B^{-1}(\bar{L} \oplus k)/(L \oplus k + U \oplus k \cap B^{-1}(\bar{L} \oplus k))$. It remains to use the relation

$$\chi'(v) U(1, v) U(1, c) = \chi'(v) \exp(2\pi i n E \oplus k(c, v)) U(1, v) U(1, c) = \chi^k(v) U(1, v) U(1, c)$$

and the lemma below.

**Lemma 2.3.3.** In the situation above

$$\det_{U \oplus k \cap B^{-1}(\bar{L} \oplus k)}(B^{-1}(U \oplus k) \cap B^{-1}(\bar{L} \oplus k)) = \det B^{-1} \cdot \frac{n^k}{d},$$

where $d = [B^{-1}(\bar{L} \oplus k) : (L \oplus k + U \oplus k \cap B^{-1}(L \oplus k))]$. 

\[ \square \]
Proof. We can write
\[
\det_{U \oplus k \cap B^{-1}(\tilde{L}^{\oplus k})} (B^{-1}(U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k})) = \det_{U \oplus k \cap B^{-1}(\tilde{L}^{\oplus k})} (U^{\oplus k} \cap L^{\oplus k}) \\
\times \det_{U \oplus k \cap L^{\oplus k}} (B^{-1}(U^{\oplus k} \cap L^{\oplus k})) \cdot \det_{B^{-1}(U \oplus k \cap L^{\oplus k})} (B^{-1}(U^{\oplus k} \cap \tilde{L}^{\oplus k})) \\
= [U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}) : U^{\oplus k} \cap L^{\oplus k}] \cdot \det B^{-1} \cdot [U^{\oplus k} \cap \tilde{L}^{\oplus k} : U^{\oplus k} \cap L^{\oplus k}]^{-1}.
\]

Now we use the formula
\[
[B^{-1}(\tilde{L}^{\oplus k}) : L^{\oplus k}] = [B^{-1}(\tilde{L}^{\oplus k}) : (L^{\oplus k} + U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}))] \\
\times [U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}) : U^{\oplus k} \cap L^{\oplus k}] \\
= d \cdot [U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}) : U^{\oplus k} \cap L^{\oplus k}].
\]

Since the lattice $B^{-1}(\tilde{L}^{\oplus k})$ is self-dual with respect to $nE^{\oplus k}$, it follows that
\[
[B^{-1}(\tilde{L}^{\oplus k}) : L^{\oplus k}] = \left[\frac{1}{n} L^\perp : L \right]^{k/2} = n^{\theta k} \cdot [L^\perp : L]^{k/2}.
\]

Together with the previous formula, this leads to
\[
[U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}) : U^{\oplus k} \cap L^{\oplus k}] = d^{-1} \cdot n^{\theta k} \cdot [L^\perp : L]^{k/2}.
\]

It remains to use the fact that
\[
[U^{\oplus k} \cap \tilde{L}^{\oplus k} : U^{\oplus k} \cap L^{\oplus k}] = [U \cap L^\perp : U \cap L]^k = [L^\perp : L]^{k/2}.
\]

Corollary 2.3.4. Assume that $U^{\oplus k} \cap B^{-1}(\tilde{L}^{\oplus k}) \subset L^{\oplus k}$ and that the line bundle $\mathcal{L}(H, \chi)$ on $V/L$ is of the form $L(1)$ for some $\Sigma_{R, e}$-structure $r \mapsto L(r)$. Then one has
\[
B^* \theta_{H^{\oplus k}, L^{\oplus k}, U^{\oplus k}}^\oplus = \zeta \cdot \det B^{-1/2} [L^\perp : L]^{-k/4} \cdot \sum_{v \in B^{-1}(\tilde{L}^{\oplus k})/L^{\oplus k}} \chi(Bv)U_{(1, v)} \theta_{nH^{\oplus k}, L^{\oplus k}, U^{\oplus k}}^\oplus.
\]

Remarks. (1) Following Shimura, let us define
\[
f_v(x) = \exp \left(-\frac{\pi}{2} H(x, x)\right) f(x)
\]
for every function $f$ on $V$. In the case when $V/L$ has many complex multiplications, Shimura [8] defined a subset $T_a(H, L, \chi) \subset T(H, L, \chi)$ consisting of functions $f$ for which $f_q(QL) \subset K'_{ab}$, where $K'_{ab}$ is the maximal abelian extension of $K'$, the reflex of the CM-field $K$ associated with $V/L$ (see [9]). It is shown in [8] that, in fact, $T_a(H, L, \chi)$ generates $T(H, L, \chi)$; more precisely, the standard basis of $T(H, L, \chi)$ multiplied by a suitable constant is a basis of $T_a(H, L, \chi)$ over $K'_{ab}$ (see [8, Prop. 2.4]). Now it follows from definition that the map $B^*$ for a matrix $B$ as above sends $T_a(H^{\oplus k}, L^{\oplus k}, \chi^{\oplus k})$ to $T_a(nH^{\oplus k}, L^{\oplus k}, B^{-1}\chi^{\oplus k})$. Our theorem gives an explicit formula for this operator in terms of standard bases of these $K'_{ab}$-linear spaces (note that $\det B \in K'$).

(2) If the line bundle $t^*_L(H, \chi)$ extends to a $\Sigma_{R,c}$-structure for some $c \in (1/2)L^\perp$, then the same simplification of theta characteristics as in the above corollary can be achieved—one just has to replace $\theta$ by $U_{(1,c)\theta}$ in formula (2.3.6).

Let us rewrite the formula (2.3.6) of Corollary 2.3.4 in the classical notation. Namely, assume that $V = \mathbb{C}^g$ and $L = \mathbb{Z}^g + \mathbb{Z}^g$, where $Z \in \mathfrak{S}_g$, $H = H_Z$ is given by $\text{Im} Z^{-1}$ (so that $L^\perp = L$), $U = \mathbb{R}^g \subset \mathbb{C}^g$, and $\chi = \chi_0(ZZ^g, \mathbb{Z}^g)$. Then the corollary can be restated as follows: if $\mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g$ has a complex multiplicity by $R$ and $L(1) = \mathcal{L}'(H_Z, \chi_0(ZZ^g, \mathbb{Z}^g))$ extends to a $\Sigma_{R,c}$-structure, then for every matrix $B = (b_{ij}) \in M_k(R)$ such that $B^* \cdot B = n \cdot \text{Id}$, $n \in \mathbb{Z}_{>0}$, and $(\mathbb{R}^g)^{\oplus k} \cap B^{-1}(L^{\oplus k}) = (\mathbb{Z}^g)^{\oplus k}$, one has

$$\exp \left( \frac{\pi}{2} \sum_{i=1}^k (Bx)^t_i \cdot (\text{Im} Z)^{-1} \cdot (Bx)_i - \frac{\pi}{2} n \sum_{i=1}^k x_i^t \cdot (\text{Im} Z)^{-1} \cdot x_i \right) \cdot \prod_{i=1}^k \theta \left( \sum_{i=1}^k b_{ij} x_j, Z \right)$$

$$= \zeta \cdot \det B^{-1/2} \cdot \sum_{v \in B^{-1}(L^{\oplus k})/L^{\oplus k}} \exp \left( \pi i \sum_{i=1}^k (Bv)^t_{i,1} \cdot (Bv)_{i,2} - \pi i n \sum_{i=1}^k v_{i,1}^t \cdot v_{i,2} \right)$$

$$\cdot \prod_{i=1}^k \left[ v_{i,1} \quad v_{i,2} \right] \left( n x_{i,1} \quad n Z \right),$$

where $x \in (\mathbb{C}^g)^{\oplus k}$, for every $y \in (\mathbb{C}^g)^{\oplus k}$ we denote by $y_i \in \mathbb{C}^g$, $1 \leq i \leq k$, the components of $y$, $y_{i,1}$, $y_{i,2} \in \mathbb{R}^g$ are the corresponding real components: $y_i = Z y_{i,1} + y_{i,2}$.

Here are examples of matrices $B$ in the case $g = 1$ for which the condition $(\mathbb{R}^1)^{\oplus k} \cap B^{-1}(L^{\oplus k}) = (\mathbb{Z}^1)^{\oplus k}$ is satisfied. If $k = 1$, then it simply means that $B = (b)$, where $b \in L$ is a primitive element of the lattice. If $k = 2$, we can take

$$B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

such that $L = a\mathbb{Z} + b\mathbb{Z}$, to satisfy this condition.
REFERENCES


