Chapter 2: Helmholtz's Equations

1.1 The Helmholtz Equation

Example 1: Open 4-dimensional Minkowski space will admit \( x^I = (c^2, x^1, x^2, x^3) \), where \( t \) is time, \( \mathbf{x} = (x^1, x^2, x^3) \) is position in real space, and \( c \) is the characteristic velocity.

Remark: (1) We will define the metric to be \( g_{\mu\nu} = (-c^2, 1, 1, 1) \). We will use Greek indices, \( \mu, \nu = 0, 1, 2, 3 \), to denote coordinates in \( 4D \) and Latin indices, \( i, j = 1, 2, 3 \), to denote coordinates in the 3-D Euclidean subspace.

Example 2: Empty space ("vacuum") supports an electromagnetic 4-vector field \( A^\nu(x) \), called the electromagnetic 4-vector potential, obeying \( \nabla \cdot A^\nu = 0 \).

Definition: The real 4-Helmholtz-Maxwell far field is

\[
F^\nu(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x).
\]

is called electromagnetic field tensor.

Remark: (2) \( \partial^\nu A^\mu(x) \) can be written in terms of a Euclidean vector field \( \phi \) and a Euclidean pseudo-vector field.
Example 1: (a) The physical field configuration $A^\mu(x)$ is determined by the action

$$S = \frac{1}{16\pi} \int d^4x \ F_{\mu\nu}(x) F^{\mu\nu}(x)$$

and minimizes the action

$$\delta S = \int d^4x \ \delta A^\mu(x) \ F^{\mu
\nu}(x) = 0$$

(b) The action is invariant under changes in the units used.

Remark: (1) The principle of least action, as is
demonstrated elsewhere. However, a search for
genuine field configurations $A^\mu(x)$ is cumbersome or impossible.

(2) $A^\mu(x)$ is the only invariance under functions.

(3) $F_{\mu\nu}(x)$ is a physical scalar $\sim$ the thing in which
\(\delta S = 0 \), but not under Galilean boosts.

(4) Maxwell did not formulate the theory like this. Advantages
of this formulation: it is not on a vacuum, easy
to quantize, analogous to other field theories (GR, perturbative physics).

(5) Heisenberg was the first unified field theory.

Example 2: (a) Noether is determined by (among other things) a 4-vector
\[ \mathcal{J}^\mu(x) \] that is parallel to $A^\mu(x)$ via

$$S = \int d^4x \ \mathcal{J}^\mu(x) A^\mu(x)$$

\[ \mathcal{J}^\mu \] is called the current of the system.

(b) The field plus its interaction with a given $\mathcal{J}^\mu(x)$ is
determined by the action $\int d^4x \ S$.

Remark: (6) This does not include the feedback from the field to the matter.
For that, one needs another pair of the electric and magnetic

\[ \mathcal{J} \] fields.
def. 2: The dual field hour (under which name) is defined as
\[ \mathcal{F}^{\mu} = \epsilon^{\mu \nu \lambda \sigma} F_{\nu \lambda} \] will \( \mathcal{F}^{\mu \nu} \) be limited

\[ \frac{\partial \mathcal{F}^{\mu}(x)}{\partial x} = 0 \] (8) proof: Problem 3.12

Remark: (8) The structure of \( \mathcal{F}^{\mu} = \text{line of products of } k^{+} \text{ metrics} \) (8).

1.6 Euler-Lagrange equations for fields

Need home work:

Lagrangian:
\[ L = g_{ik}(\phi(x,t), \phi_t(x,t), \ldots, \phi^{(k)}(x,t)) \]

action:
\[ S = \int dt L(\phi(x,t), \phi_t(x,t)) \]

Extremals:
\[ 0 = \delta S = \int dt \left( \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \phi_t} \delta \phi_t \right) \]

Further:
\[ \frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \phi_t} = 0 \]

Now consider field theory: Minkowsk space \( \mathbb{R} \) a field
\[ \phi(x) = \phi(x,t) \]

can be considered a cycle with \( t = \infty \) if we identify \( \phi(x,t) = g(x,t) \), \( \phi(x,t) = g_{ik}(x,t) \), etc. The Lagrangian now becomes:

Lagrangian density:
\[ L(\phi(x,t), \phi_t(x,t), \phi^{(k)}(x,t)) \]

relative to the spin field, in addition to time derivatives, all

Lagrangian:
\[ L = \int dt \mathcal{L}(\phi(x,t), \phi_t(x,t), \phi^{(k)}(x,t)) \]

hence the spin field
The action \( S = \int dt \int L = \int d^4x \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \)

is defined as for \( t < \infty \).

\[ 0 = \delta S = \int d^4x \left[ \frac{\partial}{\partial \phi} \delta \phi + \frac{\partial}{\partial (\partial_\mu \phi)} \delta \phi \right] \\
- \int d^4x \left[ \frac{\partial}{\partial \phi} - C \frac{\partial}{\partial (\partial_\mu \phi)} \right] \delta \phi + \delta \phi \]

\[ \Rightarrow \quad \frac{\partial}{\partial (\partial_\mu \phi)} = \frac{\partial}{\partial \phi} \quad (\dagger) \]

Remark: (1) \((\dagger)\) is the Euler-Lagrange equation for the scalar field \( \phi \). (This is the first of many, i.e., in Problem 10 for a remodeled version of the text.)

(2) Generalization to vector fields is straightforward: just add more (discrete) indices.

(3) In general, \( L \) will depend on higher gradients. \( \delta \phi \)

And (3) postulates that the Noether \( \delta \) has two gradients of \( F_{\mu\nu} \), or two gradients of \( A_\mu \).

(4) The EL eqn for fields on PDEs, in contrast to the con

1.2 The field equations

\[ \mathcal{L}(A_\mu(x)) = -\frac{1}{16\pi} F_{\mu

\[ x, \text{ the first two depend on gradients of } A_{\mu}(x), \text{ and the third one is gradient-free} \]
\[ \frac{\partial}{\partial \delta A_\mu(x)} \chi + \frac{1}{16\pi} \frac{\partial}{\partial \delta A_\mu(x)} \left( \frac{\partial A_\nu - \partial \nu A_\mu}{\partial A_\mu} \right) \left( \partial_{\nu} A_{\lambda} - \partial_{\lambda} A_{\nu} \right) \frac{1}{\sqrt{\gamma}} \gamma^\mu \gamma^\lambda \gamma^\nu \] 

\[ = \frac{1}{16\pi} \int \frac{\partial A_{\lambda}}{\partial A_{\mu}} \left[ \left( \delta^\mu_\lambda + \delta^\mu_\nu - \delta^\mu_\gamma \delta^\gamma_\nu \right) \left( \partial_{\nu} A_{\lambda} - \partial_{\lambda} A_{\nu} \right) \right] 

- \frac{1}{16\pi} \left( \frac{\partial A_{\lambda}}{\partial A_{\mu}} - \frac{\partial A_{\lambda}}{\partial A_{\nu}} \right) \left( \partial_{\nu} A_{\lambda} - \partial_{\lambda} A_{\nu} \right) + \frac{1}{16\pi} \left( \frac{\partial A_{\lambda}}{\partial A_{\mu}} - \frac{\partial A_{\lambda}}{\partial A_{\nu}} \right) \left( \partial_{\nu} A_{\lambda} - \partial_{\lambda} A_{\nu} \right) \right] 

= \frac{1}{4\pi} \left( \partial A_{\lambda} - \partial A_{\lambda} \right) \right. 

\[ \frac{1}{4\pi} \frac{\partial}{\partial \delta A_\mu(x)} \chi = \frac{1}{4\pi} \frac{\partial}{\partial \delta A_\mu(x)} \chi + \frac{1}{4\pi} \frac{\partial}{\partial \delta A_\mu(x)} \chi 

\text{EL eqn needs} \quad \frac{1}{4\pi} \frac{\partial}{\partial \delta A_\mu(x)} \chi = - \frac{1}{c} \partial \nu(x) \] 

\[ \Rightarrow \partial \nu E^{\nu}(x) = \frac{1}{c} \partial \nu(x) \] 

Remark: (1) All physical fields obey (1) 

(2) $E^{\nu}$ is defined as $A^{\nu}$, so (2) should be written as $\partial \nu E^{\nu}(x)$ for $A^{\nu}(x)$. Attempting, we can obtain (3) by the property from (1.1): 

\[ \partial \nu E^{\nu}(x) = \frac{1}{c} \partial \nu(x) \] 

with unit the length of $E^{\nu}$ in units of length of $A^{\nu}$. We can then write (2) + (3) the field $\chi$ for $F^{\mu\nu}(x)$, which is now the fundamental field.
2.1 Continuity equation for the 4-current

Proposition: The 4-current vector obeys the continuity eq.

\[ \frac{\partial}{\partial x_i} j^i(x) = 0 \]  

Proof: \[ \frac{\partial}{\partial x_i} j^i(x) = \frac{\partial}{\partial t} j^i(x) + \frac{\partial}{\partial x_i} j^i(x) = 0 \]

Remark: (1) The 4-vector \( j^i = (\rho, j^i) \) has a time-like component \( j^0 = \rho \) and three space-like components \( j^i \).

\( \rho \) is called electric charge and \( j^i \) is called electric current density.

(2) In terms of \( S \) and \( j^i \), (8) reads \( \frac{\partial}{\partial x^i} (j^i + \rho) = 0 \)

\( \implies \frac{\partial}{\partial x^i} j^i = 0 \), \( \implies \frac{\partial}{\partial x^i} j^i = \frac{2}{\partial x^i} \) \( \implies J \) is a constant of \( J \)

\[ \implies \frac{\partial}{\partial t} j^i(x, t) + \frac{\partial}{\partial x^i} j^i(x, t) = 0 \]  

(15)

is equivalent to (8).

(3) Why does \( (15) \) not apply to a spatial volume \( V \) but does apply to \( (6) \)?

\[ \frac{\partial}{\partial t} \int_V \rho d^3x \frac{\partial}{\partial x^i} j^i(x, t) = - \int_V \frac{\partial}{\partial x^i} \nabla \cdot j^i(x, t) = - \int_V \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} j^i(x, t) \]  

Define \( Q := \int_V \rho d^3x \) to be the total charge within \( V \)

\[ \implies \frac{\partial}{\partial t} Q = - \int_V \frac{\partial}{\partial x^i} j^i(x, t) \]  

\( \implies Q \) can change in or out of the volume \( V \), but the name \( \rho \) remains.
2.2 The energy-momentum tensor

\[
T^{\mu \nu}(x) = \frac{1}{4 \pi} F^\mu \nu (x) F_{\nu \lambda} (x) + \frac{i}{16 \pi} \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta} (x) F_{\lambda \kappa} (x)
\]

is called the electromagnetic energy-momentum tensor.

Remark: (1) It is not obvious what this has to do with energy and momentum, see Problem 12 for a hint, and LL for details.

Proposition: (1) \( T^{\mu \nu} \) is symmetric, \( T^{\mu \nu} = T^{\nu \mu} \)

(2) \( T^{\mu \mu} \) is traceless, \( \sum_{\mu} T^{\mu \mu} = 0 \)

Proof: (1) \( \sum_{\mu} T^{\mu \mu} = \sum_{\mu} \frac{1}{4 \pi} F_{\mu \nu} F^{\mu \nu} = \frac{1}{4 \pi} \epsilon_{\mu \nu \alpha \beta} F_{\alpha \beta} F^{\mu \nu} = \delta_{\mu \nu} \sum_{\alpha \beta \gamma \delta} F_{\alpha \beta} F^{\gamma \delta} = 0 \)

Remark: (2) \( \epsilon_{\mu \nu \alpha \beta} \) is a degenerate 4-dimensional index, corresponding to a totally antisymmetric 4-index.

2.3 The continuity equation for the energy-momentum tensor

The above of course the

Proposition: The energy-momentum tensor obeys the continuity eq.

\[
\partial_{\mu} T^{\mu \nu}(x) = 0
\]

Proof:

\[
\partial_{\mu} T^{\mu \nu}(x) = \frac{i}{16 \pi} \epsilon_{\mu \nu \alpha \beta} \left[ - \partial_{\alpha} F_{\beta \gamma} F^{\beta \gamma} + \frac{i}{2} \partial_{\beta} \delta_{\nu \alpha} F_{\rho \delta} F^{\rho \delta} \right]
\]
\begin{align*}
\frac{1}{\varepsilon^2} \left[ - (\partial_\nu F^\mu) F_\nu - F^\nu F_\nu + \frac{i}{g} \partial_\nu F^\mu F^\nu \chi \right] \\
0 = \frac{1}{\varepsilon} \left[ - (\partial_\nu F^\mu) F_\nu + \frac{i}{g} (\partial_\nu F^\mu) F^\nu \chi \right] \\
\text{Problem 17 } \Rightarrow 0 = \partial_\nu F^\mu + \partial_\chi F^\mu + \partial_\chi F^\mu \\
\frac{1}{\varepsilon^2} \left[ - (\partial_\nu F^\mu) F_\nu + \frac{i}{g} (\partial_\nu F^\mu) F^\nu \chi - \frac{i}{g} (\partial_\mu F^\nu) F^\nu \chi \right] = 0 \\
\text{Wardley: In the process of making the continuity of fields modified to} \\
\begin{aligned}
\partial_\nu \phi^\mu (x) &= -\frac{1}{\varepsilon} F_\nu^\mu (x) \phi (x) \\
\text{Proof: Problem 17}
\end{aligned}
\end{align*}

Remark: For any rank-\((n+1)\) hermit field \(F^{\nu x_1 \cdots x_n} (x)\) the continuity of \(F^{\nu x_1 \cdots x_n} (x)\) by the orbital \(\frac{\partial}{\partial \nu} \phi (x) = 0\) implies

\begin{align*}
\text{Wardley: In the process of making the continuity of fields modified to} \\
\begin{aligned}
\partial_\nu \phi^\mu (x) &= -\frac{1}{\varepsilon} F_\nu^\mu (x) \phi (x) \\
\text{Proof: Problem 17}
\end{aligned}
\end{align*}
2.4 Gauge Invariance

Let \( X(x) \) be an arbitrary vector field of space-time.

**Definition:** A transformation of the potential \( A^\mu(x) \) according to

\[
A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) - \partial^\mu X(x)
\]

is called a gauge transformation.

**Proposition:** The action from \( \{1.1 \text{ cinit} \} \) is invariant under gauge

**Proof:**

\[
F^\mu_\nu(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)
\]

\[
\rightarrow \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = \partial^\mu X(x) + \partial^\nu X(x)
\]

\[
\rightarrow F^\mu_\nu(x) \rightarrow \text{the field term is invariant}
\]

\[
S_{\text{int}} = -\frac{1}{2} \int d^4x \partial^\mu A^\nu(x) \rightarrow S_{\text{int}} + \frac{1}{2} \int d^4x \partial^\mu X(x)
\]

\[
pert = S_{\text{int}} = -\frac{1}{2} \int d^4x (\partial^\mu A^\nu(x)) X(x) = S_{\text{int}}
\]

\[
\rightarrow S_{\text{int}} \text{ is invariant}
\]

**Remark:**

(1) The potential is not unique. This is a result of the fact that

\( F^\mu_\nu \) depends only on the derivatives of \( A^\mu \).

(2) We may choose a gauge to fix a particular structure

or \( A^\mu \) ("fixing the gauge").

**Ward-Takahashi Identity:**

\[
\partial^\mu A^\nu(x) = 0 \text{ (Ward identity)}
\]

**Proof:** Using \( X \) not \( A^\mu \) it solves the PDE

\[
\partial^\mu X(x) \partial^\nu A^\mu(x) = 0
\]

\[
\rightarrow \partial^\mu A^\nu(x) = \partial^\nu A^\mu(x) - \partial^\nu \partial^\mu X(x) = 0
\]

**Remark:** \( \partial^\mu A^\nu \) is a Lorentz scalar \( \rightarrow \) the Lorentz gauge is Lorentz invariant.
Electric and magnetic fields

1.1 The field laws in herm of Euclidean vector fields

\[
\begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
e_0 & \frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} \\
e & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y} & 0 \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
-\frac{\partial}{\partial z} & \frac{\partial}{\partial y}
\end{pmatrix}
\]

\[
E = (E_x, E_y, E_z)
\]

will \( E(x) = (E_x(x), E_y(x), E_z(x)) \) a Euclidean vector field.

call \( A(x) = (A_x(x), A_y(x), A_z(x)) \) a Euclidean pseudovector field.

definitions:
1. \( E(x) \) is called the electric field, and \( A(x) \) is called the magnetic field.

2. The antisymmetric Euclidean law

\[
\begin{pmatrix}
0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial z} & \frac{\partial}{\partial x} & 0
\end{pmatrix}
\]

is called the magnetic field law.

remark:
1. \( E \cdot A = \int_A \mathbf{E} \cdot \mathbf{A} \, dA = \left( \begin{array}{c} \frac{\partial}{\partial y} \\
-\frac{\partial}{\partial x}
\end{array} \right) \cdot \left( \begin{array}{c}
\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y}
\end{array} \right) F \cdot \nu = \left( \begin{array}{c} 0 \\
-\frac{\partial}{\partial x}
\end{array} \right)
\]

1. \( A^T \) has the form

\[
A^T(x) = (\phi(x), \mathbf{A}(x))
\]

will \( \phi(x) \) a Euclidean scalar field

call \( \mathbf{A}(x) \) a Euclidean vector field.

definitions:
1. \( \phi(x) \) is called scalar potential, and \( \mathbf{A}(x) \) is called vector potential.

remark:
1. \( \mathbf{F}(x) = (\mathbf{E}(x), \mathbf{A}(x)) \) will the long chain of \( \phi(x) \) the usual derivative in \( \mathbf{A}(x) \).
1.2 Maxwell's equations

With the field given by Eq. 1.2, we have

\( E_{\perp} \square = 0 \)

\( \nabla \cdot E = 0 \)

\( \nabla \times E = 0 \)

\( \nabla \cdot B = 0 \)

\( \nabla \times B = \mu_0 J \)

**Proposition 1:** Eq. 1.2 (ee) is equivalent to

\[
\begin{align*}
\nabla \cdot \mathbf{E} (x) &= 0 \quad \text{(1)} \\
\frac{1}{c} \partial_t \mathbf{E} (x) + \nabla \times \mathbf{B} (x) &= 0 \quad \text{(2)}
\end{align*}
\]

**Remark:** (1) Then one has homogeneous PDEs for the field components \( E_x, E_y, E_z \).

**Now consider the \( E \) component:**

\( \partial_{\perp} E_{\perp} = \frac{\mu_0}{c} j_{\perp} \)

\( \nabla \cdot E = 0 \)

\( \frac{\partial}{\partial E_{\perp}} E_{\perp} = \frac{\mu_0}{c} j_{\perp} \rightarrow \frac{\partial}{\partial \mathbf{E}} \mathbf{E} = \frac{\mu_0}{c} \mathbf{j} \rightarrow \nabla \cdot \mathbf{E} = \frac{\mu_0}{c} \mathbf{j} \)

\( \nabla \times E = 0 \)

\( \frac{\partial}{\partial E_{\perp}} E_{\perp} = \frac{\mu_0}{c} j_{\perp} \rightarrow -\frac{1}{c} \partial_t E_x + \partial_y E_z - \partial_z E_y = \frac{\mu_0}{c} j_x \)

**Proposition 2:** Eq. 1.2 (ee) is equivalent to

\[
\begin{align*}
\nabla \cdot E (x) &= \frac{\mu_0}{c} j (x) \quad \text{(3)} \\
\frac{1}{c} \partial_t \mathbf{E} (x) + \nabla \times \mathbf{B} (x) &= \frac{\mu_0}{c} \mathbf{j} (x) \quad \text{(4)}
\end{align*}
\]
Units: \( \text{gauss} \) = \( \text{C/m}^2 \), \( \text{magnetomotive force (mmf)} = \text{ampere-turns} \)

\[
\begin{align*}
[\mathbf{B}] &= \text{j}^1 \mu_0 \mathbf{H}^2, \\
[\mathbf{H}] &= \text{j}^2 \mu_0 \mathbf{E}^2, \\
[\mathbf{E}] &= \text{j}^3 \mu_0 \mathbf{B}^2.
\end{align*}
\]

In SI units:

\[
\begin{align*}
[\text{mmf}] &= C, \\
[\mathbf{B}] &= 1 \text{T}, \\
[\mathbf{H}] &= 1 \text{A/m}, \\
[\mathbf{E}] &= 1 \text{V/m}, \\
[\mathbf{B}] &= 1 \text{T/m}, \\
[\mathbf{H}] &= 1 \text{A/m}.
\end{align*}
\]

\[
\begin{align*}
\mathbf{B} &= \frac{\mathbf{H}}{\mu_0}, \\
\mathbf{E} &= \frac{\mathbf{B}}{\mu_0}, \\
\mathbf{H} &= \frac{\mathbf{E}}{\mu_0},
\end{align*}
\]

\[
\begin{align*}
\mathbf{E} &= \nabla \times \mathbf{B}, \\
\mathbf{B} &= \nabla \times \mathbf{H}, \\
\mathbf{H} &= \nabla \times \mathbf{E}.
\end{align*}
\]

\[
\begin{align*}
\mathbf{E} &= \mathbf{E}_0 + \nabla \phi, \\
\mathbf{B} &= \mathbf{B}_0 + \nabla \mathbf{A},
\end{align*}
\]

When

\[
\begin{align*}
\varepsilon_0 &= 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2, \\
\mu_0 &= 4\pi \times 10^{-7} \text{ N/A}^2
\end{align*}
\]

and

\[
\begin{align*}
\varepsilon_0 &= 1/\varepsilon_0 = 1.11 \times 10^{-12} \text{ C}^2/\text{Nm}^2.
\end{align*}
\]
work: 1. Eqs (11)-(1) are called Maxwell's eqs. Their solutions determine physical field configurations for given charge and current densities.

2. Eqs (11)-(1) are equivalent to Eqs. (5), (6) and \( \mathbf{F} \) is a \( \mathbf{E} \) field, \( \mathbf{E} \) is an electric vector field. \( \mathbf{D} \) is a \( \mathbf{D} \) field, \( \mathbf{D} \) is an electric vector field. The Lorenz invariance of (1)-(4) is far from obvious, whereas the Lorentz invariance of (5), (6) is.

2.2 Discussion of Maxwell's equations

Under a localized charge density \( \mathbf{J} \) with a volume \( V \) will be in (V).

\[ \int V \mathbf{E} \cdot d\mathbf{V} = \oint C(x,t) = Q(t) \cdot \mathbf{E} \] holds where

\[ \oint C(x,t) = \text{flux of } \mathbf{E} \text{ thru the surface} \] (V)

(0) This is called Gaunt's law

1. Electric fields are the sources of electric fields.

\[ = \oint C(x,t) = \text{flux of } \mathbf{E} \text{ thru the surface} \] (V)

\[ = \oint C(x,t) = \text{flux of } \mathbf{E} \text{ thru the surface} \] (V)

Remark: 2. The magnetic field has no sources! Equivalent statement: There is no magnetic charge (no magnetic monopole).

2. No magnetic field can be in a localized region of \( \mathbf{B} \) field. This originally manifests itself in the original field eqs:

\[ \partial \mathbf{E} \cdot \mathbf{F} = \frac{\mathbf{G}}{\mathbf{G}}, \quad \partial \mathbf{E} \cdot \mathbf{F} = 0 \]
\[ \mathbf{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \]

**Remark:**
1. This is called **Faraday's law of induction.**
2. \( \text{lin.} \Rightarrow \mathbf{E} \) \( \Rightarrow \nabla \times \mathbf{E} \) \( \Rightarrow \frac{d}{dt} \mathbf{D} \Rightarrow \frac{\partial \mathbf{D}}{\partial t} \)
3. Time derivative of flux of \( \mathbf{D} \) \( \text{through the surface} S \)

\[ \mathbf{E} = \mathbf{\nabla} \varphi \quad \text{or} \quad \mathbf{E} = \mathbf{\nabla} \varphi \]

**Remark:**
4. This is called **Ampère-Maxwell law.**

5. \( \text{lin.} \Rightarrow \mathbf{H} \) \( \Rightarrow \nabla \times \mathbf{H} \) \( \Rightarrow \mu_0 \mathbf{J} \)

6. Ampère's law

7. \( \mu_0 \) is the **magnetic permeability of vacuum**

8. For static fields, we have Ampère's law

\[ \mathbf{\nabla} \cdot \mathbf{D} = \frac{\rho}{\varepsilon} \quad \text{in static fields} \]

The displacement current was later added by Maxwell.
1.4 Relation between fields and potentials

\[ E^i = - \nabla^0 i = - \partial^0 A^i + \partial^i A^0 \]
\[ = - \partial_0 A^i - \partial_i A^0 = - \frac{1}{c} \partial_t A^i - \partial_i \varphi \]

\[ \Rightarrow \quad \overline{E}(\overline{x}, t) = - \nabla \varphi(\overline{x}, t) - \frac{1}{c} \partial_t \overline{A}(\overline{x}, t) \]

Remark: (1) In vacuo, both \( \overline{A} \) and \( \varphi \) determine \( \overline{E} \).

(2) General before: \( A^i \rightarrow A^i - \partial^i X \quad \Rightarrow \quad \varphi - \varphi - \frac{1}{c} \partial_t X \)
\[ \overline{A} \rightarrow \overline{A} + \overline{X} \]
\[ \rightarrow \overline{E} \rightarrow \overline{E} + \nabla^i \partial_i X - \frac{1}{c} \partial_t \overline{X} = \overline{E} \quad \overline{E} \text{ is wave function} \]

\[ F_{ik} = (\partial_k A^l - \partial_l A^k) = - (\nabla \times \overline{A}) \]
\[ = - \nabla^2 \overline{A} \quad \text{and} \quad \Delta c \overline{E} \]
\[ \Rightarrow \quad \overline{B}(\overline{x}, t) = \nabla \times \overline{A}(\overline{x}, t) \]

Remark: (3) General before, \( \nabla \times (\nabla \varphi) = 0 \)
3.5. Charge in electromagnetic fields

Attitude to far: Field ions determine fields for given charges and fields.

Question: For given fields, what is their influence on a point dipole?

At a point particle will charge e be at point \( \mathbf{q}(t) \) will induce

\[
\mathbf{E}(r) = \frac{\mathbf{q}(t)}{|r|}
\]

\[
\mathbf{D}(r) = \varepsilon_0 \mathbf{E}(r)
\]

\[
\mathbf{P}(r) = \frac{\varepsilon_0}{2} \mathbf{D}(r)
\]

\[
\mathbf{B}(r) = \mu_0 \mathbf{H}(r)
\]

\[
\mathbf{H}(r) = \frac{\mathbf{J}(r)}{\mu_0}
\]

**Ex. 1.1**

\[
\Delta \mathbf{t} = - \frac{1}{c} \int d^4 \xi \cdot \mathbf{j}(\xi) \cdot \mathbf{A}(\xi)
\]

\[
= - \frac{c}{e} \int d^4 \xi \cdot \mathbf{A}(\xi) \cdot \mathbf{F} = - \frac{c}{e} \int d^4 \xi \cdot \mathbf{A}(\xi) \cdot \mathbf{F}
\]

Now consider the Lagrangian of the point particle, let \( \Delta t (\mathbf{q}, \mathbf{v}, t) \),

which is related to \( \Delta \mathbf{t} \) via \( \Delta \mathbf{t} = \int d^4 \xi \cdot \mathbf{F} \)

\[
\Delta \mathbf{t} = \int d^4 \xi \cdot \mathbf{F}
\]

**Remark:**

(1) Then on the vector ad vector potentials from PATH 629

2. § 1.2 example 1

(2) § 1.1 exercise 2 is unimportant and somewhat trivial.

(3) Let us treat the Lagrangian \( \mathcal{L}(\mathbf{F}, \mathbf{F}) \) for a free particle.

Field is on Lorentz invariant \( \rightarrow \) No need to Einstein's \( \mathcal{L}(\mathbf{F}, \mathbf{F}) \).

Note: Believo (c) is only approx. for 10^16 c/s.)
\[ \text{Problem 2.10} \]

**Coulomb's Law:**
\[ F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \]

where \( q_1 \) and \( q_2 \) are the charges and \( r \) is the distance between them.

**Electric Field:**
\[ E = \frac{F}{q} \]

**Electric Potential:**
\[ V = \int E \cdot dx \]

**Magnetic Force:**
\[ F = q(E \times v) \]

**Magnetic Field:**
\[ B = \frac{\mu_0 I}{4\pi r^2} \]

**Magnetic Potential Energy:**
\[ U = \frac{1}{2} q(B \cdot v) \]

**Maxwell's Equations:**
\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times H = J + \frac{\partial D}{\partial t} \]

**Divergence Theorem:**
\[ \int_{V} \left( \nabla \cdot \mathbf{F} \right) \, dV = \int_{S} \left( \mathbf{F} \cdot \mathbf{n} \right) \, dS \]

**Gauss's Law:**
\[ \int_{V} \nabla \cdot \mathbf{D} \, dV = \int_{S} \mathbf{D} \cdot \mathbf{n} \, dS = \int_{S} \mathbf{E} \cdot \mathbf{n} \, dS \]

**Faraday's Law:**
\[ \int_{C} \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot d\mathbf{A} \]

**Ampère's Law:**
\[ \int_{C} \mathbf{H} \cdot d\mathbf{r} = \frac{1}{\mu_0} \int_{V} \left( \nabla \times \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \, dV \]

**Biot-Savart Law:**
\[ \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I \, d\mathbf{l} \times \mathbf{r}}{r^3} \]

**Magnetic Induction:**
\[ \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \]

**Potential Energy in an Electric Field:**
\[ U = \frac{1}{2} q^2 E^2 \]

**Potential Energy in a Magnetic Field:**
\[ U = \frac{1}{2} q^2 B^2 \]

**Kinetic Energy:**
\[ K = \frac{1}{2} m v^2 \]

**Total Energy:**
\[ E = U + K \]

**Conservation of Energy:**
\[ \frac{d}{dt} \left( \frac{1}{2} m v^2 + \frac{1}{2} q^2 \mathbf{E}^2 \right) = -q \mathbf{E} \cdot \mathbf{F} \]

**Conservation of Momentum:**
\[ \frac{d}{dt} \left( m \mathbf{v} + q \mathbf{E} \right) = \mathbf{F} \]

**Lorentz Force:**
\[ \mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

**Mass:**
\[ m = \frac{E}{c^2} \]

**Charge:**
\[ q = \frac{E}{c} \]

\( E, B \) are Electric and Magnetic fields,
\( v \) is the velocity of an object.
\[ E = \frac{1}{2} \gamma (\text{for any particle, just run over all}) \]
\[ \int d^3 r \int d^2 E \cdot \varepsilon(x,t) \cdot \varepsilon(x',t') = \varepsilon(x,0) \cdot \varepsilon(x',0) \]

where \( \gamma \) is the particle of the particle.

Under some relativistic postulate for EMaxwell (the

example also holds in the relativistic case, in Eq. 17)

\[ E = \frac{c}{v} \varepsilon \quad \text{and the energy} \]

\[ \frac{d}{dt} E = m \varepsilon \cdot \frac{d \varepsilon}{dt} = \varepsilon \cdot \frac{d \varepsilon}{dt} = \varepsilon \cdot \dot{\varepsilon} \]

Now suppose \( P \) moves over all of space:

\[ \frac{d}{dt} \int d^3 x \varepsilon(x,t) + \int d^3 x \int d^2 \varepsilon(x,t) = - \int d^3 x \int d^2 \varepsilon(x,t) = - \int \varepsilon \cdot \dot{\varepsilon} \]

\[ \int \varepsilon \cdot \dot{\varepsilon} = 0 \quad \text{in } \varepsilon \cdot \dot{\varepsilon} \text{ at } \infty \]

\[ \frac{d}{dt} (\varepsilon + E) = 0 \quad \text{with } \varepsilon = \int d^3 x \varepsilon(x,t) \]

\[ U \text{ must be the field energy, i.e., the energy of the particle plus the energy of the field must be conserved} \]

\[ U(x,t) \text{ must be the energy density of the field} \]

(4) Suppose over a finite volume \( U \) Energy may change due to a energy flux across the volume boundary

\[ \dot{\mathbf{P}} \text{ shall be interpreted as the energy flux density of the field} \]

(5) The normal \( \mathbf{N} \)-parts of the continuity Eq. for \( \mathbf{E} \)

\[ \partial_t \mathbf{N} = - \mathbf{E} \cdot \mathbf{N} \]

express the fact that the energy flux density is also conserved.
4.1 Physical interpretation of a Lorentz boost

Consider two inertial frames, $cs$ and $cs'$, let $cs'$ move with respect to $cs$ with a relative velocity $\mathbf{V} = (V, 0, 0)$.

Let $x, t \rightarrow \tilde{x}, \tilde{t}$ be the Lorentz transformation of the coordinates.

$$
\tilde{x} = ct \cos \phi + x \sin \phi
\quad \text{and} \quad
\tilde{t} = ct \sin \phi + x \cos \phi
$$

Writing the units $\tilde{\tilde{c}} = \tilde{c}$ as usual $\tilde{\tilde{c}} = c$. Then $\tilde{\tilde{c}} = 0$

$$
\Rightarrow x \cos \phi = -ct \sin \phi
\quad \Rightarrow \quad V = x \cos \phi = -ct \sin \phi
$$

$$
\Rightarrow \frac{\sin \phi}{\cos \phi} = \frac{Vc}{\tilde{c}c} = \frac{Vc}{\sqrt{1 - V^2/c^2}} \quad \text{and} \quad \cos \phi = \frac{1}{\sqrt{1 - V^2/c^2}}
$$

Remark: (1) For $c \to \infty$ we recover the Galilean boost

$$
\tilde{x} = x + Vt, \quad \tilde{t} = t
$$

$A$ Lorentz boost along the $x$-axis is given by

$$
\Delta \mathbf{v} = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
\sin \phi & -\cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

with

$$
\cos \phi = \frac{1}{\sqrt{1 - V^2/c^2}}, \quad \sin \phi = \frac{V}{\sqrt{1 - V^2/c^2}} \quad \text{and} \quad \tilde{\tilde{c}} = \frac{1}{\sqrt{1 - V^2/c^2}}
$$
4.2 Transformation of $E$ and $B$ under a Lorentz boost

Write the field in $\mathbb{R}^\nu$; i.e., the field in $\mathbb{R}^\nu$

\[ \widehat{F}^\nu = \delta^\nu_\lambda F^\lambda \]

Now let $\delta^\nu_\lambda$ in a Lorentz boost.

\[ \Rightarrow \widehat{F}^\nu = (\delta^2 D^T)^\nu_\lambda F^\lambda \]

\[ = \begin{pmatrix}
    \omega \phi & \nu \phi & \nu \phi & 0 & 0 \\
    \nu \phi & \omega \phi & \nu \phi & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\[ = \begin{pmatrix}
    -E_x \omega \phi & -E_x \nu \phi & -E_y \nu \phi & -E_y \omega \phi & -E_y \nu \phi \\
    E_x \nu \phi & -E_x \omega \phi & E_y \nu \phi & -E_y \omega \phi & -E_y \nu \phi \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[ = \begin{pmatrix}
    0 & 0 & 0 & E_x \nu \phi & E_y \nu \phi \\
    0 & 0 & 0 & E_x \nu \phi & 0 \\
    0 & 0 & 0 & E_y \nu \phi & 0 \\
    0 & 0 & 0 & E_y \nu \phi & E_z \nu \phi
\end{pmatrix}
\]

Remark: (1) The field is now Lorentz invariant.

Their limit ionization is guaranteed. (2) Reflects the limit ionization of $\mathbb{R}$'s eqs. with

\[ \begin{pmatrix}
    E_x \\
    E_y \\
    E_z
\end{pmatrix}
\]

\[ \Rightarrow \begin{pmatrix}
    \tilde{E}_x = E_x \\
    \tilde{E}_y = E_x \nu \phi + E_y \nu \phi \\
    \tilde{E}_z = E_z \nu \phi - E_y \nu \phi
\end{pmatrix}
\]

\[ \begin{pmatrix}
    \tilde{x} \\
    \tilde{y} \\
    \tilde{z}
\end{pmatrix}
\]

\[ \Rightarrow \begin{pmatrix}
    \tilde{x} = \tilde{x} \\
    \tilde{y} = \tilde{y} \nu \phi - E_y \nu \phi \\
    \tilde{z} = \tilde{z} \nu \phi + E_y \nu \phi
\end{pmatrix} \]
Work (2) \[ \text{let } V/c \ll 1 \text{ ad loop has to } O(V/c) . \]
\[ \Rightarrow \text{let } \phi = 1 , \text{ with } \phi = V/c \]
\[ \Rightarrow \tilde{E} = E - \frac{1}{c} \vec{V} \times \vec{a} + O(V/c^2) , \quad \tilde{B} = \tilde{B} + \frac{1}{c} \vec{V} \times E + O(1/V/c^2) \]

(2) \[ \text{let } \vec{E} = 0 , \text{ i.e., no } E \text{-field i.e.} \] \[ \Rightarrow \tilde{E} = -\frac{1}{c} \vec{V} \times \vec{a} \Rightarrow \text{has } \tilde{B} \text{ flux i.e. ad } E \text{-field, as} \]
\[ \text{let } \vec{V} \rightarrow 0 ! \]

4.2 Work identities

For the field laws \[ F \rightarrow \text{we can form two work identities:} \]

\[ \mathcal{F}(5) : = \frac{1}{2} \vec{E} \cdot \vec{F}_{ij} , \quad \mathcal{F}(11) : = \frac{1}{8} \epsilon_{ijk} \epsilon_{pqr} \vec{F}_{ij} \vec{F}_{pq} \]

Remark: (5) \[ \mathcal{F}(5) \text{ is a scalar } \Rightarrow \mathcal{F}(5) = \mathcal{F}(5) \text{ must be all unimportant.} \]

\[ \mathcal{F}(11) \text{ is a pseudoscalar } \Rightarrow |\mathcal{F}(11)| = \text{ work i.e. all unimportant.} \]

\[ \mathcal{F}(11) \rightarrow \mathcal{F}(5) = \frac{1}{2} \left( \frac{\epsilon_{i j} \vec{E} \cdot \vec{F}_{i j}}{\epsilon_{k l} \vec{E} \cdot \vec{F}_{k l}} \right) = \vec{E}^2 - \frac{1}{c} \vec{E} \cdot \vec{B} \]

\[ \epsilon_{i j} \epsilon_{k l} = \epsilon_{i k} \epsilon_{j l} + \epsilon_{i l} \epsilon_{j k} \quad \vec{E} \times \vec{F} = \frac{1}{2} (\delta_{j i} \delta_{k l} - \delta_{i j} \delta_{k l}) \vec{F}_{i j} \vec{F}_{k l} \]

\[ \mathcal{F}(11) = \frac{1}{8} \left[ 2 \epsilon_{012} F_{01} F_{23} + 2 \epsilon_{012} F_{02} F_{13} = \frac{1}{4} \left( \epsilon_{012} F_{01} F_{23} + \epsilon_{012} F_{02} F_{13} \right) + 2 \epsilon_{012} F_{01} F_{23} \right. \]

\[ + \epsilon_{012} F_{01} F_{23} + \epsilon_{012} F_{02} F_{13} + \epsilon_{012} F_{02} F_{13} \]

\[ + (1 \times 6 = 24 \text{ other terms}) \right] \]
\[
\frac{1}{4} \left[ - E_x u_x - E_y u_y - E_z u_z \right] \times \gamma = - \frac{E^2}{2}
\]

Proposition: The field combination

\[
\mathcal{F}^{(1)} = E^2 - \gamma^2, \quad \mathcal{F}^{(2)} = E \cdot \mathbf{u}
\]

\text{Remark: (1)}: This is obviously true since the theory is linear.

\text{Remark: (2)}: If the active medium has higher conductivity \( \gamma \), it would not be true.

\text{Remark: (3)}: A field theory that leads to linear field \( \gamma \) is called Gaussian.
Wolley 1: Let \( \tilde{E}^{(1)}(x_0) \), \( \tilde{G}^{(1)}(x_0) \) be soln for \( g^{(1)}(x_1) \), \( h^{(1)}(x_1) \) in \( \mathbb{R} \), and let \( x_k \in \mathbb{R} \) be arbitrary s.t. \( k \neq 0 \) and \( x_k \neq x_0 \).

\[
\tilde{E}(x) = \int dh \lambda(x) \tilde{E}^{(1)}(x), \quad \tilde{G}(x) = \int dh \lambda(x) \tilde{G}^{(1)}(x)
\]

are soln for \( g(x) = \int dh \lambda(x) g^{(1)}(x), \quad h(x) = \int dh \lambda(x) h^{(1)}(x) \).

Proof: Assume in prop 1 to \( k = 1, \ldots, N \) and let \( N \to \infty \).

Wolley 2: The most general soln of \( H \)'s eqs. is obtained as

\[
\tilde{E}(x) = \tilde{E}^{(1)}(x) + \tilde{E}^{(I)}(x_1), \quad \tilde{G}(x) = \tilde{G}^{(1)}(x) + \tilde{G}^{(I)}(x_1)
\]

where \( \tilde{E}^{(1)}, \tilde{G}^{(1)} \) are the most general soln in vacuum
(i.e., for \( x = x_0 \)) and \( \tilde{E}^{(I)}, \tilde{G}^{(I)} \) is a particular soln of the eqs. in the presence of \( \tilde{J} \).

Proof: Let \( \tilde{E}, \tilde{G} \) be any soln for \( \tilde{J} \), and let \( \tilde{E}^{(I)}, \tilde{G}^{(I)} \)
be a particular soln. Prop 1 \( \Rightarrow \)

\[
\tilde{E}^{(1)} = \tilde{E} - \tilde{E}^{(I)}, \quad \tilde{G}^{(1)} = \tilde{G} - \tilde{G}^{(I)}
\]

is a soln for \( \tilde{J} = 0 \).

Since \( \tilde{J} \), if \( \tilde{E}^{(1)} \) is a soln for \( \tilde{J} = 0 \), \( \tilde{G}^{(1)} \) is some soln for \( \tilde{J} = 0 \), then \( \tilde{E} = \tilde{E}^{(1)} + \tilde{E}^{(I)}, \quad \tilde{G} = \tilde{G}^{(1)} + \tilde{G}^{(I)} \) is a soln for \( \tilde{J} = 0 \).

5.2 Upper solutions: If \( H \)'s eqs. need of total fields \( \tilde{E} \) and \( \tilde{G} \),

we obtain fields \( \tilde{E} \) and \( \tilde{G} \).

However, it sometimes is convenient to find upper solutions
and treat the total part.
Proposition: Let \( E, \bar{E} \) be complex solutions for complex \( S, \bar{S} \). Then \( E, \bar{E} \) are solutions for \( S, \bar{S} \).

Proof:
\[
\nabla E = \nabla S \bar{E} + i \nabla S \bar{E} = \bar{E} \nabla S + i \bar{E} \nabla S
\]
\[
\Rightarrow \nabla E = \bar{E} \nabla S \\
\Rightarrow \nabla (\bar{E} E) = \bar{E} \nabla S - i \bar{E} \nabla S
\]

etc.

Remark: (1) This gives the kernel of linearity.

Theorem: Let \( E, \bar{E} \) be complex solutions for real (i.e., physical) \( \nabla S \). Then \( \Re E, \Re \bar{E} \) are also solutions for \( \nabla S \).

Proof: \( \text{Let } \Re E, \Re \bar{E} \text{ be solutions for } \nabla S \).

Remark: (2) In this case, \( \Re E, \Re \bar{E} \) are solutions in the sense of kernels (i.e., \( \Re \nabla S = \Re \nabla S = 0 \)).