9. **Time-like and space-like intervals**

Consider two points \((ct, x^1, x^2, x^3)\) and \((ct, y^1, y^2, y^3)\) in Minkowski space. The interval between the two points is called *time-like* if

\[
c^2(t_x - t_y)^2 > (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 ,
\]

and *space-like* if

\[
c^2(t_x - t_y)^2 < (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 .
\]

Show that in interval that is time-like or space-like in some inertial frame is also time-like or space-like in any other inertial frame. (This reflects the invariance of the speed of light.)

(2 points)

10. **Special Lorentz transformations**

Consider the Minkowski space \(M_4\).

a) Show that the following transformations are Lorentz transformations:

i) \(D^\mu_\nu = \begin{pmatrix} 1 & 0 \\
0 & R^i_j \end{pmatrix} \equiv R^\mu_\nu \) (rotations)

where \(R^i_j\) is any Euclidian orthogonal transformation.

ii) \(D^\mu_\nu = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\
\sinh \alpha & \cosh \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu_\nu \) (Lorentz boost along the \(x\)-direction)

with \(\alpha \in \mathbb{R}\).

iii) \(D^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu_\nu \) (parity)

iv) \(D^\mu_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu_\nu \) (time reversal)

b) Let \(L\) be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of \(L\), and so are the Lorentz boosts defined in part a) ii).

c) Let \(I^\mu_\nu = \delta^\mu_\nu\) be the identity transformation. Show that the sets \(\{I, P\}, \{I, T\}, \) and \(\{I, P, T, PT\}\) are subgroups of \(L\).

(4 points)
11. **General Lorentz transformations**

Let $D$ be a general Lorentz transformation.

a) Show that $|D^0_0| \geq 1$, and that $(D^0_1)^2 + (D^0_2)^2 + (D^0_3)^2 = (D^1_0)^2 + (D^2_0)^2 + (D^3_0)^2$.

b) Let $L_{++} = \{D \in L; \det D > 0, D^0_0 > 0\}$. (This is called the set of proper orthochronous Lorentz transformations, and one can show that it is a subgroup of $L$.) Show that any Lorentz transformation can be written as an element of $L_{++}$ followed by either $P$, or $T$, or $PT$. It thus suffices to study $L_{++}$.

c) Show that any element of $L_{++}$ can be written as a spatial rotation $R(\Phi, \Theta, \Psi)$ followed by a Lorentz boost $B(\alpha)$ followed by a rotation about the 3-axes followed by a rotation about the 2-axis. In a symbolic notation:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_2(\phi)R_3(\theta) & 0 \\ 0 & 0 & R(\Phi, \Theta, \Psi) \end{pmatrix}$$

$L_{++}$ is thus characterized by six parameters: 3 Euler angles $\Phi, \Theta, \Psi$, the boost parameter $\alpha$, and two additional rotation angles $\phi$ and $\theta$.

(7 points)