Transformation problems

1. Consider the line $\ell$ given by $y = 3x + 2$. In this problem we will find a formula for the transformation that reflects the plane in the line $\ell$.

   (a) Let $m$ be the line that is parallel to $\ell$ and passes through the origin. Write down the equation for $m$. Using the method we learned in class (and practiced in problem 3 of HW#2), find the matrix $A$ for which $F(x) = Ax$ is reflection in the line $m$.

   (b) Pick any point $(a, b)$ on $\ell$, and let $T$ be the translation $T(x, y) = (x - a, y - b)$. It is helpful to use vector notation and write

   $$ T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - a \\ y - b \end{bmatrix}. $$

   Notice that this sends $\ell$ to $m$. What is the formula for $T^{-1}$?

   (c) The reflection $R_{\ell}$ is given by

   $$ R_{\ell} = T^{-1} \circ R_m \circ T. $$

   Use this to help you determine a formula

   $$ R_{\ell}(x) = Bx + c $$

   where $B$ is an appropriate $2 \times 2$ matrix and $c$ is an appropriate vector.

2. Let an “eye” be placed at point $E = (0, 0, 10)$. Let $M$ be the plane $y = 5$ and $N$ be the plane $z = 0$. Let $F$ be the projective transformation that maps $M$ to $N$, along radial lines emanating from the point $E$. See the picture below.

   Determine a formula for what the transformation $F$ does to a point $P = (x, 5, z)$, by following the steps below:
(a) Write down a parametric equation for the line $\overrightarrow{EP}$: the line consists of all points of the form

$$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} + t \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

where you fill in the question marks.

(b) Find the value of $t$ that makes the third coordinate equal to zero. Plugging this back in to your expression from (a), what are the coordinates for the point $F(P)$?

(c) For what values of $x$ and/or $z$ is $F(x,5,z)$ undefined?

(d) Give an example showing that $F$ does not preserve distances. That is, find two points $P$ and $Q$ on $M$, compute the distance $d(P,Q)$ and the distance $d(F(P),F(Q))$, and see that they are not the same.

Euclidean geometry problems

3. Given: $U$ is the centroid of $\triangle AMG$ and $V$ is the centroid of $\triangle MBG$. Point $J$ is the midpoint of $AG$, $K$ is the midpoint of $MG$, and $L$ is the midpoint of $BG$. Let $D$ be the intersection of $\overrightarrow{UV}$ with $AG$, and $F$ be the intersection of $\overrightarrow{UV}$ with $BG$. Determine the ratio $\frac{GF}{GB}$ and prove that your answer is correct. What can you say about $\frac{GD}{GA}$? [Hint to help organization: Define $u$ to be the length of $LB$. Determine some other lengths in the diagram as certain multiples of $u$.]

\begin{center}
\includegraphics[width=\textwidth]{triangle3.png}
\end{center}
4. Given: $M$, $N$, and $P$ are the midpoints of $AB$, $AC$, and $BC$, respectively. Also, $U$, $V$, $W$, $X$, $Y$, and $Z$ are the centroids of the six “little triangles”. Let $F_1$ be the intersection of $UV$ with $GB$, and let $F_2$ be the intersection of $XW$ with $GB$.

(a) Use problem #3 to explain why $F_1 = F_2$. That is, prove that $\overrightarrow{UV}$, $\overrightarrow{GB}$, and $\overrightarrow{XW}$ are concurrent.

(b) Now refer to the enhanced diagram we have below (points $D$, $E$, and $F$ are given to us by applications of part (a)). Prove that the area of $\triangle DEF$ is $4/9$ of the area of triangle $\triangle ABC$. 
5. Given: the things that look like squares are squares. To prove: the areas of the four shaded triangles are all equal.

![Diagram of squares and triangles]

6. This is a problem in 3-dimensional geometry. Given: $A$, $B$, and $C$ are in the indicated plane, $RBS$ is a straight line, $RB = SB$, $AB \perp RS$, and $\angle CAR = \angle CAS$. To prove: $\angle ACR = \angle ACS$ and $BC \perp RS$.

![Diagram of 3D geometry]

**NOTE:** Later in this assignment you will type up your solution to this problem in LaTeX. If you don’t want to write it out by hand, that is fine. Just put a little note telling the grader to look at the LaTeX printout.
7. Given: $ABCDEF$ is a hexagon inscribed in a circle, $AB \parallel ED$ and $BC \parallel EF$. Prove that $CD \parallel AF$. 

![Hexagon inscribed in a circle with parallel lines](image)
**LaTeX problems**

On this assignment you will learn about three new things:

1. matrices
2. controlling sizes for parentheses and brackets.
3. displayed math

**Matrices:** The following LaTeX code typesets a matrix:

```latex
\begin{bmatrix}
5 & x & 17 \\
y & 8 & e^2 \\
\end{bmatrix}
```

This has to be in math mode or displayed math mode. The `&` signs say ‘go the the right one entry’, where as the `\` signs say ‘go to the next row and start over’. So the above code produces the matrix

\[
\begin{bmatrix}
5 & x & 17 \\
y & 8 & e^2 \\
\end{bmatrix}
\]

The code for a $3 \times 1$ matrix would look like this:

```latex
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
```

whereas the code for a $1 \times 3$ matrix would look like

```latex
\begin{bmatrix}
x & y & z \\
\end{bmatrix}
```

**Size of delimiters:** If we write

\[
f\left(\frac{3}{5}\right) = \frac{7}{8}
\]

then it looks funny. We need to be able to tell LaTeX to use bigger parentheses, or bigger brackets. The commands that do this come in four sizes:

\[
\big, \ Big, \bigg, \ Bigg.
\]
The code
\[ f \bigg( \frac{3}{5} \bigg) = \frac{7}{8} \]
produces the output
\[ f \left( \frac{3}{5} \right) = \frac{7}{8} \]
whereas the code
\[ f \bigg( \frac{3}{5} \bigg) = \frac{7}{8} \]
produces the output
\[ f \left( \frac{3}{5} \right) = \frac{7}{8} \]

Note that the ‘\\’ and ‘\’ commands are for display-math mode. This is like regular math mode, but produces the output on its own line and nicely centers it; also, things tend to be printed just a little bit bigger in display-math mode.

The \texttt{big/Big/bigg/Bigg} ‘size commands’ can be used with parentheses, brackets, set brackets, the ‘|’ sign, and various other delimiters. Some people like to write ‘\Bigg’ and ‘\Biggr’ to distinguish between left and right delimiters, as this sometimes makes the spacing work look a little nicer, but it is very subtle.

Note that to make a set bracket you have to type “\\{” because LaTeX thinks that \{ by itself is an internal grouping, not text. For example, the text
\[ \bigl\{ A \in \text{M}_{2\times 2}(\mathbb{R}) \bigm| A^2 = A \biggr\} \]
produces the output
\[ \left\{ A \in \text{M}_{2\times 2}(\mathbb{R}) \bigm| A^2 = A \right\} \]

Notice the use of “\\,” here. This command says “add a tiny bit more space”. Without this, the math would look a bit too cramped around the “such that” bar in the set descriptor.

8. Take a look at the files “hw3\_model.tex” and “hw3\_model.pdf” on the course website. Download these. Also download the file “FIG395-1D.pdf”, which is needed for LaTeX to compile the document.

(a) In the “hw3\_model.pdf” file you will see eight numbered lines on the first page. These are produced by the “\item” commands you see in the .tex file. Each “item” has a line of displayed math following it, and your job will be to alter the displayed math. Do not mess around with the “\item” commands themselves!

Equations (1)–(5) all give formulas that don’t look very nice because the wrong size delimiters were used. Make use of the \texttt{big/Big/bigg/Bigg} commands and fix each of these lines so that they look as nice as possible. You will have to play around to
decide which size looks best.

(b) In line (6), change this so that it shows a $2 \times 3$ matrix multiplying a $3 \times 1$ column vector, to give a $2 \times 1$ column vector.

(c) In line (7), fill in the correct formula for the inverse of a $2 \times 2$ matrix.

(d) In line (8), write a matrix using the “pmatrix” environment rather than the “bmatrix” environment. Notice the difference. It will be important that “pmatrix” is written in both your \begin and \end statements.

(e) In line (9), do the same thing as (8) but use the “matrix” environment instead of the “bmatrix” environment. Notice the difference. I never really use the “matrix” environment; I tend to always use bmatrix or pmatrix.

(f) On page 2 of the LaTeX document, type up your proof for problem #6 on this homework.