Transformation problems

1. The counterclockwise rotation through $90$ degrees about the origin is given by $\rho(x) = Ax$ where $A$ is the matrix
$$
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
$$
Take this as a given.

Let $F$ be the counterclockwise rotation through $90$ degrees about the point $(2,5)$. Using the same method as in the last homework assignment, determine a formula $F(x) = Bx + c$ for appropriate $B$ and $c$ that you produce.

Check: your formula should yield that $F(2,5) = (2,5)$ and $F(0,0) = (7,3)$.

[Hint: Use that $F = T^{-1} \circ \rho \circ T$ where $T$ is an appropriate translation.]

Before we get to the next question we need a little discussion. Let $G$ be the translation $G(x,y) = (x+5,y+7)$. If we want to transform the point $(1,3)$ we just put it into $G$ and get $G(1,3) = (6,10)$. Easy. But suppose we want to transform the whole circle $x^2 + y^2 = 1$?

This is harder, as we cannot just “apply $G$” to this equation. Here is what you do instead:

1. Introduce new coordinates $a$ and $b$ to stand for the image of a point $(x,y)$: that is,
$$
\begin{bmatrix}
a \\
b
\end{bmatrix} = G\left(\begin{bmatrix}x \\ y\end{bmatrix}\right).
$$
In our particular case $a = x + 5$ and $b = y + 7$.

2. Now invert the above equation to get
$$
\begin{bmatrix}
x \\
y
\end{bmatrix} = G^{-1}\left(\begin{bmatrix}a \\ b\end{bmatrix}\right).
$$
In practice, this means solving for $x$ and $y$ in terms of $a$ and $b$. In our particular example we get $x = a - 5$ and $y = b - 7$.

3. Now plug in these formulas for $x$ and $y$ into the equation $x^2 + y^2 = 1$ that we started with. Of course we get
$$(a - 5)^2 + (b - 7)^2 = 1.$$ 

Hopefully you recognize this as the equation for a circle of radius one centered at $(5,7)$; it is exactly the result of applying $G$ to our original circle.

The letters $a$ and $b$ were introduced so that we could perform the above manipulations, but often we take them away at the end. It is more common to say that $G$ transforms the circle $x^2 + y^2 = 1$ into the circle $(x - 5)^2 + (y - 7)^2 = 1$. 
Let $\Omega$ denote the parabola $y = x^2$, and let $\ell$ be the line $y = 3x$. Note that $(0,0)$ and $(3,9)$ are on both $\Omega$ and $\ell$.

(a) Let $T$ be the translation $T(x, y) = (x + 5, y + 10)$. Use the procedure described above to find the equation of the transformed parabola $T(\Omega)$. Write your equation in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A$, $B$, $C$, $D$, $E$, and $F$ are constants that you determine. Check that $(5,10)$ and $(8,19)$ satisfy your equation.

(b) Let $m$ be the line $2x + 3y = 7$. Find the matrix that represents the reflection $R_\ell$ (you have done this many times now) and then use it to find the equation for the reflected line $R_\ell(m)$. Write your equation in the form $Ax + By = C$. [Hint: You will need to find the inverse of the matrix. You can either do this the usual way, or use the fact that $R^{-1}_\ell$ is easy to describe].

(c) Find the formula for the reflected parabola $R_\ell(\Omega)$. Again, write your equation in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Check that the points $(0,0)$ and $(3,9)$ satisfy your equation.

**Projective geometry problems**

3. (a) Start with the cubic $y = x^3$. When we put this into the projective plane it becomes the set of all points $[x : x^3 : 1]$. Determine the point at infinity on this cubic.

(b) Now do the same thing for the graph of $y = e^x$. This time there are two points at infinity: find them.

(c) Finally, do the same thing for the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$. (Start by solving for $y$ in terms of $x$). There are again two points at infinity: find them.

(d) For the hyperbola considered in (c), what are the slopes of the two asymptotes? How do you know?

[Warning: On this problem you will have to remember from basic calculus how to compute certain limits. We might have to review some of this.]
Euclidean geometry problems

NOTE: For questions 5 and 6 you must write up your proof using LaTeX. You get 5 points extra credit if you also write up your proof of #4 using LaTeX. You can use the file “hw4 eg.tex” on the course website as a starting point. If you have trouble including the diagrams, you can always use the command \textbackslash vspace{2in} to tell LaTeX to leave some space (in this case, 2 inches) and then you can draw the diagram in by hand.

4. Given: \( M, N, \) and \( P \) are the midpoints of \( AB, AC, \) and \( BC, \) respectively. Also, \( U, V, \) \( W, X, Y, \) and \( Z \) are the centroids of the six “little triangles”. Remember that you can use any facts proven in previous homeworks.

\begin{center}
\includegraphics[width=\textwidth]{triangle_diagram.png}
\end{center}

(a) Prove that the area of \( \triangle DZU \) is 1/16 of the area of \( \triangle DEF. \)

(b) Prove that the area of hexagon \( UVWXYZ \) is \( \frac{13}{36} \) of the area of \( \triangle ABC. \) (Yay! This is what we have been working towards.)
5. This is a problem in 3-dimensional geometry. Given: $A$, $B$, and $P$, and $Q$ are in the indicated plane, $RPS$ and $AQB$ are straight lines, $RS \perp PA$, $RS \perp PB$, $SP = RP$. To prove: $RS \perp PQ$.

![Diagram](image1)

6. Given: $\angle ABC$ is a right angle, and $ABDE$ and $ACFG$ are squares. To prove: $BG = CE$ and $BG \perp CE$.

![Diagram](image2)
LaTeX problems

As you have discovered, one of the difficulties with LaTeX is debugging the error messages. For one thing, the error messages usually do not say anything very helpful. The most common sources of errors are

- Using math symbols without telling LaTeX to go into math mode
- Telling LaTeX to go into math mode but then forgetting to tell it to come out of math mode: every dollar sign must have a mate, and every \[ must have a corresponding \].
- Having mismatched brace symbols, e.g. having a { without a corresponding } or vice versa.

Finding errors can be a pain, but here is one tip that sometimes helps. When LaTeX hits an error it spits out something about what/where it thinks the problem is, gives you the ‘? ’ sign, and waits. Sometimes you will be able to look at the error message and determine exactly where the problem is in your file: then you go fix it and recompile. But sometimes you can’t see where the problem is; in this case, try hitting the “return” (or “enter”) key once or twice (sometimes multiple times). Usually LaTeX will then skip over this point as best it can and continue compiling. Then you can look in the Previewer and see where things start to look crazy, and that usually tells you where the problem is. This process doesn’t always work (sometimes LaTeX refuses to skip over the problem), but it often does.

In the next problem you will get some practice with debugging LaTeX code. The process takes some getting used to.

7. Download the file “hw4_model.tex” from the course website. I have intentionally put several errors into this file, so when you open it in LaTeX and try to compile you will see error messages right away. This is intentional, so don’t panic.

Find the place in the tex file that says “\bf 1.". Look through this paragraph of text and try to find the things that need to be fixed. The error message(s) produced by LaTeX might give you some clues. Fix the errors and recompile until all the errors disappear.

In order to help you focus on one debugging issue at a time, I have temporarily put the “\end{document}” command right after the first debugging issue. So when you are done with item #1 you will see only that small part show up in the Preview window. At that point, move the \end{document} command to right before the “\{ bf 3.\}” part and try to compile again. Now debug this next bit, and continue with this process until you are all done. There are six debugging exercises in all.

8. In LaTeX an “environment” is a grouping of code that begins with a \begin command and ends with an \end command. On the last assignment you learned about the bmatrix and pmatrix environments.

Two other useful environments are “itemize” and “enumerate”. These are for making lists: itemize is for bulleted lists, and enumerate is for lists where the different items are each given some kind of sequential label (1,2,3 or a,b,c or A, B, C, and so forth).
The assignment on this question is simple. Read through the LaTeX code in the “hw4_model.tex” and make sure you understand how the itemize and enumerate environments work. The particular brand of “enumerate” that I like to use requires an extra package. Look through the frontmatter of the LaTeX code until you find the spot where we tell LaTeX to use this package.

You don’t need to hand in anything for this problem.