Math 253
Homework due Wednesday, February 17

1. Section 8.3: 14, 21, 26.
2. Section 8.4: 26, 32, 33.
3. Section 8.5: 11, 13, 14, 15, 17, 33, 34.
4. Section 8.6: 4, 5, 8, 9, 11, 14, 15, 17.

5. Let \( f(x) = \frac{x}{x^2 + 16} \).
   
   (a) Use Mathematica to graph this function.
   (b) Use Mathematica to find a series approximation to this function; compute this out to at least degree 20.
   (c) Plot a graph showing both the function \( f(x) \) and its degree 3 Taylor polynomial. Where do the functions seem close, and where do they seem to diverge from each other?
   (d) Repeat part (c) for the degree 9 Taylor polynomial.
   (e) It is natural to guess that by taking a large enough degree, we can make the Taylor polynomial approximate \( f(x) \) over any given interval. This is false. Explain why, by analyzing the radius of convergence of the power series. Start with
   \[
   \frac{x}{x^2 + 16} = \frac{1}{16} \cdot \left( \frac{x}{\frac{x^2}{16} + 1} \right),
   \]
   then proceed as in problems 3–10 of Section 8.6.

6. (a) How many terms of the series \( \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^{1.5}} \) do we need to add in order to be within \( 10^{-4} \) of the limit? Find the answer using the Alternating Series Estimation Theorem from Section 8.4.
   (b) Use Mathematica to approximate the limit to six decimal places, then use Mathematica to compute the sum of the first \( N \) terms where \( N \) is the number you found in (a). Check that these are within \( 10^{-4} \) of each other. [Hint: Mathematica will give greater accuracy if you input \( \frac{3}{2} \) instead of 1.5].

7. (a) Determine a power series approximation for \( \int \frac{1}{1+x^2} \, dx \).
   (b) Suppose we want to compute \( \int_{0}^{0.1} \frac{1}{1+x^2} \, dx \). How many terms of the power series will we need if we want to get within \( 10^{-5} \) of the correct answer? (Use the Alternating Series Estimation Theorem).
(c) Use Mathematica to compute a decimal approximation to the integral, as follows:

\[
N\left[\text{Integrate}\left[\frac{1}{1+x^3},\{x,0,1/10\}\right],20\right]
\]

[Note that it is important to use 1/10 instead of 0.1: try both and see why.]

(d) Use Mathematica to calculate the approximation using the number of terms of the power series you found in (b). Check that this approximation is within \(10^{-5}\) of the answer in (c).

As soon as people learned about convergent series, it became very exciting to compute as many of them as possible. For example, what is

\[
1 - \frac{1}{4} + \frac{1}{5} - \frac{1}{8} + \frac{1}{9} - \frac{1}{12} + \frac{1}{11} - \cdots.?\]

We know this converges by the Alternating Series Test, but we are not told what it converges to. The values of these infinite series are like little treasures left by the universe for us to discover. Anytime someone figured out a new one, it was like finding a little gold nugget. In the following two problems you will explore a particular case of this, determining the limits for a certain class of series.

8. (a) Explain why \(\sum_{k=1}^{\infty} \frac{k}{10^k}\) converges.

(b) Use Mathematica to find a decimal approximation to \(\sum_{k=1}^{\infty} \frac{k}{10^k}\). Find the answer to 50 decimal places.

(c) You should notice that your answer to (b) is a repeating decimal. Convert this repeating decimal to a fraction using the algorithm we learned back in Section 8.2. If necessary, use Mathematica to reduce your fraction to lowest terms. The correct answer has two-digit numbers in the numerator and denominator.

(d) Repeat parts (b) and (c) for \(\sum_{k=1}^{\infty} \frac{k}{14^k}\) and \(\sum_{k=1}^{\infty} \frac{k}{(-2)^k}\). Do part (b) for \(\sum_{k=1}^{\infty} \frac{k}{(-6)^k}\), and think about doing (c) (but don’t actually do it).

9. Expanding on the previous question, we are going to find a general formula for \(\sum_{k=1}^{\infty} kx^k\).

(a) Start with the equation \(\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\). Differentiate both sides.

(b) Use your answer to (a) to find a formula for \(\sum_{k=1}^{\infty} kx^{k-1}\).

(c) Use your answer to (b) to find a formula for \(\sum_{k=1}^{\infty} kx^k\).

(d) What values of \(x\) does the formula in (c) apply to? Explain.

(e) Check your answer to (c) by plugging in \(x = \frac{1}{10}\) and \(x = \frac{1}{4}\) and comparing to your answers in #8 above. Do they agree?