Basic Facts

In our exploration of “Advanced Geometry” it will take too long to get going if we try to develop everything starting just with Euclid’s Axioms. For this reason, I want to use a slightly more extensive list of axioms—which I will call “Basic Facts”, so that we don’t get confused.

It’s a little strange, but we will try to keep the “Advanced Geometry” component of the course separate from the “Euclidean geometry” component. This works because you really already know the Euclidean geometry facts, just maybe not from the ancient Greek perspective.

BF 1. (SSS) Three sides determine a triangle up to congruence.
BF 2. (SAS) Two sides and the included angle determine a triangle up to congruence.
BF 3. (ASA) Two angles and the included side determine a triangle up to congruence.
BF 4. If two triangles are similar, then the ratios of the corresponding sides are all the same.
BF 5. Suppose two lines are crossed by a transversal. Then if the lines are parallel, the corresponding angles are all the same. Likewise, if two corresponding angles are the same, the lines are parallel.
BF 6. The whole is the sum of the parts.
BF 7. Through two points there is exactly one line.
BF 8. On a ray there is exactly one point a given distance from the endpoint.
BF 9. A line segment can be extended to an infinite line.
BF 10. Any line segment has a midpoint.
BF 11. Any angle can be bisected.
BF 12. Given a line $\ell$ and a point $P$, there is exactly one line which passes through $P$ and is perpendicular to $\ell$.
BF 13. Given a line $\ell$ and a point $P$, there is a line which passes through $P$ and is parallel to $\ell$.
BF 14. If two lines are parallel to a third, they are parallel to each other.
BF 15. The area of a rectangle is the base times height.
Theorems

**Theorem 1**: When two lines cross, adjacent angles add up to $180^\circ$, and vertical angles are equal.

**Theorem 2**: Suppose that two lines $\ell$ and $m$ are crossed by a transversal.

(a) If $\ell$ and $m$ are parallel, then both pairs of alternate interior angles are equal. If at least one pair of alternate interior angles are equal, then $\ell$ and $m$ are parallel.

(b) If $\ell$ and $m$ are parallel, then each pair of interior angles on the same side of the transversal adds up to $180^\circ$. If at least one pair of interior angles on the same side of the transversal adds up to $180^\circ$, then $\ell$ and $m$ are parallel.

(c) If $\ell$ and $m$ are parallel, then each pair of exterior angles on the same side of the transversal adds up to $180^\circ$. If at least one pair of exterior angles on the same side of the transversal adds up to $180^\circ$, then $\ell$ and $m$ are parallel.

**Theorem 3**: The angles of a triangle add up to $180^\circ$.

**Theorem 4**: If triangle $ABC$ and $DEF$ have $\angle A = \angle D$ and $\angle B = \angle E$, then they also have $\angle C = \angle F$.

**Theorem 5**: Two sides of a triangle are equal if and only if the opposite angles are equal.

**Theorem 6**: In a parallelogram, opposite sides are equal and opposite angles are equal.

**Theorem 7**: In the following figure, if $DE \parallel BC$ then $\frac{AD}{DB} = \frac{AE}{EC}$.

**Theorem 8**: In the above picture, if $D$ is the midpoint of $AB$ and $DE \parallel BC$ then $E$ is the midpoint of $AC$ and $DE = \frac{1}{2}BC$.

**Theorem 9**: In the previous picture, suppose we are given that $D$ is the midpoint of $AB$ and $E$ is the midpoint of $AC$. Then $DE \parallel BC$.

**Theorem 10**: Let $\ell$ be the perpendicular bisector of $AB$. Then all points on $\ell$ are equidistant from $A$ and $B$; moreover, if a point $P$ is equidistant from $A$ and $B$ then $P$ must lie on $\ell$.

**Theorem 11**: The area of a triangle is one-half of the base times the height. The area of a parallelogram is the base times the height.

**Theorem 12**: In a triangle, the perpendicular bisectors of the sides are concurrent.

**Theorem 13**: The altitudes of a triangle are concurrent.

**Lemma 13.2**: In a parallelogram, the diagonals bisect each other.

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Lemma 13.6: Given the picture below where the two lines inside the triangle are medians, one has \( MX = \frac{1}{3}BM \).

\[ \begin{array}{c}
  \text{A} \\
  \text{B} \\
  \text{C}
\end{array} \quad \begin{array}{c}
  \text{M} \\
  \text{N}
\end{array} \quad \begin{array}{c}
  \text{X}
\end{array} \]

Theorem 14: The angle bisectors of a triangle are concurrent.

Theorem 15: The medians of a triangle are concurrent.

Theorem 16 (The Pythagorean Theorem): In \( \triangle ABC \), if \( \angle A = 90^\circ \) then \( BC^2 = AB^2 + AC^2 \).

Theorem 17 (Hypotenuse-Leg): Suppose given two right triangles \( \triangle ABC \) and \( \triangle A'B'C' \), with \( \angle A = \angle A' = 90^\circ \). Then if \( AB = A'B' \) and \( BC = B'C' \), the two triangles are congruent.

Theorem 18: A quadrilateral \( ABCD \) is cyclic if and only if \( \angle A + \angle C = 180 \) (or equivalently, \( \angle B + \angle D = 180 \)).

Theorem 19: Let \( \mathcal{C} \) be a circle with center \( O \), and let \( \ell \) be a line that is tangent to \( \mathcal{C} \) at point \( P \). Then \( OP \) is perpendicular to \( \ell \).

Theorem 20 (Inscribed Angle-Arc Theorem): Let \( \mathcal{C} \) be a circle. Any two inscribed angles that cut out the same arc of \( \mathcal{C} \) must be equal.

Theorem 21: Let \( \mathcal{C} \) be a circle and \( P \) be a point outside \( \mathcal{C} \). Let \( \ell \) be a line through \( P \) that intersects \( \mathcal{C} \) in two points \( A \) and \( B \), and let \( \ell' \) be another line through \( P \) that intersects \( \mathcal{C} \) in the two points \( A' \) and \( B' \). Then \( PA \cdot PB = PA' \cdot PB' \).

Corollary 22: Let \( \mathcal{C} \) be a circle and \( P \) be a point outside of \( \mathcal{C} \). Let \( \ell \) be a line through \( P \) that intersects \( \mathcal{C} \) in two points \( A \) and \( B \), and let \( \ell' \) be a line through \( P \) that is tangent to \( \mathcal{C} \) at point \( T \). Then \( PA \cdot PB = PT^2 \).

Theorem 23: Let \( \mathcal{C} \) be a circle and \( P \) be a point inside \( \mathcal{C} \). Let \( \ell \) be a line through \( P \) that intersects \( \mathcal{C} \) in two points \( A \) and \( B \), and let \( \ell' \) be another line through \( P \) that intersects \( \mathcal{C} \) in the two points \( A' \) and \( B' \). Then \( PA \cdot PB = PA' \cdot PB' \).

Definition: Let \( \mathcal{C} \) be a circle with center \( O \) and radius \( R \), and \( P \) be any point. Then define \( \text{Power}_{\mathcal{C}}(P) = (PO)^2 - R^2 \). This is called the power of \( P \) with respect to circle \( \mathcal{C}_1 \).

Theorem 24: Let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be two circles, with centers \( O_1 \) and \( O_2 \), respectively. Then the set of points having the same power with respect to \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) forms a line that is perpendicular to \( O_1O_2 \). (This is called the radical axis of the two circles).

Theorem 25: If a triangle has two sides of length \( a \) and \( b \) and \( \theta \) is the measure of the angle where they meet, then the area of the triangle is \( \frac{1}{2}ab\sin\theta \).

Theorem 26 (Law of Sines): Suppose given triangle \( \triangle ABC \), and write \( a = BC \), \( b = AC \), \( c = AB \). Then

\[
\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)}.
\]
Theorem 26 (Law of Cosines): Suppose given triangle $\triangle ABC$, and write $a = BC$, $b = AC$, $c = AB$. Then
\[ c^2 = a^2 + b^2 - 2ab \cos(\angle C). \]

Theorem 27 (Ceva’s Theorem): Suppose given a triangle $\triangle ABC$ together with points $L$ on $\overrightarrow{AB}$, $M$ on $\overrightarrow{BC}$, and $N$ of $\overrightarrow{AC}$. Then the lines $\overrightarrow{AM}$, $\overrightarrow{BN}$, and $\overrightarrow{CL}$ are concurrent if and only if
\[ \frac{AL}{LB} \cdot \frac{BM}{MC} \cdot \frac{CN}{NA} = 1. \]

Theorem 28 (Theorem of Menelaus): Suppose given a triangle $\triangle ABC$ together with points $L$ on $\overrightarrow{AB}$, $M$ on $\overrightarrow{BC}$, and $N$ of $\overrightarrow{AC}$. Then the points $L$, $M$, and $N$ are collinear if and only if
\[ \frac{AL}{LB} \cdot \frac{BM}{MC} \cdot \frac{CN}{NA} = -1. \]
**Definition 1:** Two triangles $\triangle ABC$ and $\triangle DEF$ are similar if $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

**Definition 2:** A parallelogram is a quadrilateral in which each pair of opposite sides is parallel.

**Definition 3:** Let $A$, $B$, and $K$ be three collinear points. Define the signed ratio

\[
\frac{\overrightarrow{AK}}{\overrightarrow{KB}} = \begin{cases} 
\frac{AK}{KB} & \text{if } K \text{ is between } A \text{ and } B, \\
-\frac{AK}{KB} & \text{if } K \text{ is not between } A \text{ and } B.
\end{cases}
\]