Math 395: Guide to writing proofs

(1) First of all, the proof should *look* nice. It should be written in a neat and orderly way that makes it easy to read. Some tips related to this:

(a) Do not try to write the proof on the same sheet where the homework problems were assigned. If you do this, we won’t even look at it. Use fresh paper.

(b) Use lined paper. If you have messy writing then writing your proof on unlined paper is going to make it look even *more* messy. Also, using lined paper can help you establish a nice linear flow to the proof.

To be fair, if you have neat handwriting and write nicely structured proofs then it is fine for you to use unlined paper if you prefer. But if you write messy proofs and use unlined paper on top of it then we are going to take off for that, because you are making our task more difficult.

(c) The first line of your proof should start on the left margin of your paper. If it doesn’t, we are not even going to read the proof. Do not draw a diagram and then start the proof to the right of the diagram. Do not start the proof in the middle of the page and then draw arrows showing us how it continues to the top and sides, etc.

(d) Each line of the proof should follow the preceding one, just like you were writing an essay. When you come to the end of the page, go to a new page. Do not draw arrows telling us how the proof should be read all over the page—if you find yourself needing to do that, rewrite the proof.

Sometimes when you are re-reading a proof you will find that you would like to insert a line in the middle to help the explanation. In this case, it is okay to write the line in the margin and draw an arrow showing where it should go in the proof. But do not do this more than once per page.

(e) It helps to organize your proof into paragraphs and sections, as appropriate. It might be useful to write an outline on scratch paper of what you think the main steps of the proof are. When you are writing the proof, skip a line after completing each main step. See the example at the end of this document.

(2) If your proof refers to a diagram (and almost all geometry proofs do) then make sure the diagram is on the same sheet of paper as your proof. Do not just refer to a diagram that was given in the homework assignment—redraw the picture so that it appears on the same page as your proof. If your proof refers to a diagram that is not right in the same place as your proof, we will not even read it.

(3) Use connecting words in the proof. The most common connecting words in math proofs are: *By, Since, Because, Then, If, Assume, Suppose, Therefore, Thus, Hence.* Use them! Some people like to use the symbol ∴ for “therefore”, and an upside down version of this symbol for “we know”. We are not going to allow you to do this anymore. Please use English words and phrases in your proofs.
(4) In mathematics, the chain of symbols “$P \Rightarrow Q$” is shorthand for the statement “If $P$ is true, then $Q$ is true.” The statement says nothing at all about whether $P$ or $Q$ is actually true. A common mistake is to use $P \Rightarrow Q$ as shorthand for the statement “Because $P$ is true, we also know $Q$ is true”. Note that this last statement does say something about the truth of $P$ and $Q$. When presenting a proof verbally to someone, it is okay to mix these things up because it is usually clear from context which version is meant. But this mixing is an abuse which is not permitted in written proofs.

Do not write “$P \Rightarrow Q$” if what you mean is “Since $P$ is true, we conclude $Q$.” Use English words and write the full statement out.

Similarly, the chain of symbols “$P \Leftrightarrow Q$” is shorthand for the two statements $P \Rightarrow Q$ and $Q \Rightarrow P$. This means that the statements $P$ and $Q$ are logically equivalent: if one is true (or false) then the other is true (or false). However, $P \Leftrightarrow Q$ says nothing about whether $P$ or $Q$ is actually true. Do not write “$P \Leftrightarrow Q$” if what you mean is “$P$ is true, therefore we know that $Q$ is true.”

(5) Be careful about the order of the vertices when writing statements about triangles or polygons. If you write $\triangle ABC \cong \triangle XYZ$ then it must be the case that $AC = XZ$ and $\angle BAC = \angle YXZ$, etc. Do not write $\triangle ABC \cong \triangle XYZ$ when you mean $\triangle ABC \cong \triangle XZY$.

(6) Give citations for steps that deserve citations. It is somewhat subjective what kinds of things are “obvious” and what things are not, but this is something that hopefully you have built up a feeling for by now. I will try to keep track of things that we agree do not need a citation; here are the ones I have thought of so far:

(a) Suppose given the situation of the picture below:

If you know that $DE$ is parallel to $BC$ then you are welcome to say that $\triangle ABC \cong \triangle ADE$ without giving further explanation (you do not need to go through the proof each time). You are also welcome to say that the ratio $\frac{AD}{AE}$ is equal to the ratio of the heights of the two triangles, without having to give a reference for this.

(b) Similarly, given the situation
where $DE$ is parallel to $BC$ you are welcome to say something like $\frac{DX}{BY} = \frac{AE}{AC}$ without giving further explanation (a complete explanation would just be that $\frac{DX}{BY} = \frac{AX}{AY}$ using $\triangle ADX \simeq \triangle ABY$, and $\frac{AX}{AY} = \frac{AE}{AC}$ using $\triangle AXE \simeq \triangle AYC$).

(c) If you know $\ell \parallel \ell'$ and line $m$ is perpendicular of $\ell$, you can say that $m$ is perpendicular to $\ell'$ without further explanation.

**Point system for grading proofs:**

Each proof will be worth 10 points, divided into 3 “idea” points and 7 “style” points. The “idea points” will be awarded based on a rough judgement of how much of the problem you seemed to understand; however, there is the proviso that if the proof is written in a confusing way we are not going to spend a lot of time trying to decode exactly how far you got and will just give a 0 or 1 on the idea points.

For the style points, deductions will be made according to the following rough scheme:

- Not using a relevant connecting word: -1
- Error in use of $\Rightarrow$ or $\Leftrightarrow$: -2
- Error in order of vertices of a triangle: -2
- Not giving a citation: -1 or -2, depending on the situation.

For multiple occurrences of the same mistake, points can be deducted for each violation—but the grader can put a cap on this at his or her discretion. Points will also be deducted for insufficient explanation of a step and confusing structure, but in both cases the number of points deducted will depend on the situation.
Sample problem:
Given: $M$, $N$, and $Q$ are the midpoints of the sides of $\triangle ABC$. The point $X$ is the foot of the altitude drawn from $B$ onto $AC$. Prove that $M$, $X$, $N$, and $Q$ all lie on a common circle.

Proof #1. Let $Z$ be the intersection of $BX$ and $MQ$, and let $\alpha = \angle BAX$. We start by proving that $Z$ is the midpoint of $BX$, and then use this to prove that $\angle QMX = \alpha$.

Since $M$ and $Q$ are the midpoints of their respective sides, we know from Theorem 9 that $MQ \parallel AC$. Then applying Theorem 8 to $\triangle BAX$ gives that $Z$ is the midpoint of $BX$.

Since $MZ \parallel AX$ and $AX$ is perpendicular to $BX$, we know that $MZ$ is perpendicular to $BX$. Then $\triangle BMZ \cong \triangle XMZ$ by SAS (using side $MZ$, $\angle MZB = \angle MZX = 90^\circ$, and $ZB = ZX$). Thus, $\angle BMZ = \angle ZMX$.

Since $MQ \parallel AX$ we know $\angle BMQ = \angle BAX = \alpha$ by BF5. Therefore $\angle ZMX = \alpha$ using the conclusion of the previous paragraph. Similarly, Theorem 9 gives that $NQ \parallel BA$ and so $\angle QNC = \angle BAX = \alpha$ by BF5. Then $\angle QNA = 180 - \alpha$.

We now find that $\angle QMX + \angle QNX = \alpha + (180 - \alpha) = 180$, and so Theorem 18 gives that $QMXN$ is a cyclic quadrilateral. This means that $Q$, $M$, $X$, and $N$ all lie on a common circle.

\[\Box\]

This proof is fine, and would be full credit.
Proof #2. First, \( \angle BME = \angle BAX \). \( \angle BEM = \angle BXA = 90^\circ \) by Theorem 4. So \( \triangle BME \cong \triangle BAX \) since all three angles are equal. Therefore

\[
\frac{BE}{EX} = \frac{BM}{MA} = 1
\]

by Theorem 7. Thus, \( BE = EX \).

So \( \triangle BEM \cong \triangle XEM \) by SAS, since \( \angle BEM = \angle XEM = 90 \), \( ME = ME \), and \( BE = EX \).

We know \( QN = \frac{1}{2}BA \) by Theorem 8. \( M \) is the midpoint of \( BA \) \( \Rightarrow \) \( BM = MA \). So \( BM = \frac{1}{2}BA = QN \).

We know \( BM = MX \), so \( MX = QN \). Therefore \( \angle MQN = \angle QMX \). We know \( \angle MQN + \angle QNX = 180^\circ \) by Theorem 2. So \( \angle QMX + \angle XNQ = 180 \), and therefore \( M, X, N, \) and \( Q \) all lie on a common circle by Theorem 18.

Grading for Proof #2:

- First line: what is \( E \)? We have to figure out by reading further (and only from context) that \( E \) is the intersection of \( MQ \) and \( BX \). \(-2\)
- Still on the first line: no reference for why \( \angle BME = \angle BAX \). \(-1\)
- Second sentence: “Theorem 4” is not a correct explanation for the fact that the two angles are 90. \(-2\)
- Third sentence: this is mostly okay, but why are all three angles equal? \(-1\)
- Next few lines are okay, until we get to an incorrect use of \( \Rightarrow \) in the third paragraph. \(-2\)
• “We know $BM = MX$...” Why is this true? It comes from $\triangle BEM \cong \triangle XEM$. If it had been stated right after the congruence was proven, that would be clear enough and basically okay. But here, it is very hard to follow. $-1$

• The line “Therefore $\angle MQN = \angle QMX$” doesn’t make sense to me, and includes no citation to help me understand why this would be true. $-2$

Final style points for this problem $0/7$ (actually, the style score is negative, but we won’t count negative points).

For the idea points, it looks like Proof #2 had a lot of the main ideas. But the “Therefore...” statement is a big gap, and this is really the crucial step. So probably we would give 2/3 on the idea points, although one could make a case for 1/3.

Final score for Proof #2: 2/10.