6.6

a.

Consider a circular toroidal current distribution with mean radius $a$ and $N$ turns, with a small uniform cross sectional area $A$ and current $I$ flowing around it with a charge $Q$ at it’s center. By the elementary application of amperian loops, and neglecting the spatial extent of the interior relative to the radius of the torus, the magnetic field inside the torus is seen to be

$$\vec{B}(\vec{x}) = \begin{cases} 
\pm \frac{IN\mu_0}{2\pi a} \hat{\phi} & \text{inside the torus} \\
0 & \text{else} 
\end{cases}$$

Where the $\pm$ depends on the direction of current flow. Recall that the momentum in the fields is given by the equation

$$P_{field} = \mu_0 \epsilon_0 \int_V \mathbf{E} \times \mathbf{H} \, d^3x$$

(1)

The $E$ field is given by

$$\mathbf{E}(\vec{x}) = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

(2)

We can express the radial unit vector and the azimuthal unit vector by

$$\hat{r} = (\cos(\phi) \sin(\theta), \sin(\phi) \sin(\theta), \cos(\theta))$$

(3)

$$\hat{\phi} = (-\sin(\phi), \cos(\phi), 0)$$

(4)

so that

$$\mathbf{E} \times \mathbf{H} = \pm \frac{IN}{2\pi a} \frac{Q}{4\pi\epsilon_0 a^2} (-\cos(\theta) \cos(\phi), \cos(\theta) \sin(\phi), \sin(\theta)) \approx \pm \frac{IN}{2\pi a} \frac{Q}{4\pi\epsilon_0 a^2} (\theta, \theta, 1)$$

(5)

When we multiply by $\mu_0 \epsilon_0$ and integrate over the volume, which is approximately $2\pi a A$, we get the result for the $z$ component,

$$(P)_z = \pm \frac{INQA\mu_0}{4\pi a^2}$$

(6)

And the other components approximately zero, since all positive values of $\theta$ are compensated for with negative values of $\theta$ when an integral across the entire cross-sectional area is performed. The approximation given is

$$P_{field} = \frac{1}{c^2} \mathbf{E}(0) \times \mathbf{m}$$

The total magnetic moment is zero, but if we consider the individual contributions of each loop, and sum them, we get

$$P_{field} = \pm N \frac{1}{c^2} \frac{Q}{4\pi\epsilon_0 a^2} IA \mp \frac{INQA\mu_0}{4\pi a^2}$$

Beauty.
b. Let $Q = 10^{-6}C \equiv 6 \times 10^{12}$ electronic charges, $I = 1.0 A$, $N = 2000$, $A = 10^{-4} m^2$ and $a = .1 m$. Then, at the toroid, we have

$$|E|(a) = \frac{10^{-6} C}{4\pi (8.854 \times 10^{-12} m^{-3} kg^{-1} s^4 A^2)(.01 m^2)} = 898774 V \quad (7)$$

$$|B|(a) = \frac{1A \times 2000 \times (1.256 \times 10^{-6})}{2\pi (.1 m)} = 4 \times 10^{-3} T \quad (8)$$

$$\langle P \rangle_z = \frac{200 \times (10^{-6} C) \times (10^{-4} m^2) \times (1.25 \times 10^{-6})}{4\pi(.1m)^2} = 2 \times 10^{-12} N \cdot s \quad (9)$$

For comparison, a $10\mu g$ insect flying at $.1 m/s$ has a momentum of $p = 10^{-7} N \cdot s$. So, it is a few orders of magnitude smaller, even than this.

6.9

Consider a uniform, isotropic medium described by permittivity $\epsilon$ and $\mu$. In general, for a non-dispersive, linear medium the following equations apply

$$u = \frac{1}{2}(E \cdot D + B \cdot H) = \frac{1}{2}(\epsilon(E)^2 + \mu(B)^2) \quad \text{eqn (6.106)}$$

$$S = E \times H \quad \text{eqn (6.109)}$$

$$g = \frac{1}{c^2}(E \times H) = \mu \epsilon E \times H \quad \text{eqn (6.118)}$$

$$T_{ij} = \epsilon(E_i E_j + \frac{1}{\epsilon \mu} B_i B_j - \frac{1}{2}(E^2 + \frac{1}{\epsilon \mu} B^2)\delta_{ij}] = [\epsilon E_i E_j + \mu H_i H_j - \frac{1}{2}(\epsilon E^2 + \mu H^2)\delta_{ij}] \quad \text{eqn (6.120)}$$

Where we have modified the equations in the book by substituting $\epsilon (\mu)$ for $\epsilon_0 (\mu_0)$. These are all locally defined quantities, so the only change that needs to be made is to make $\epsilon$ and $\mu$ position-dependent quantities and not pull them out of $D$ and $H$. So the functional form of the above stays the same, if we absorb $\epsilon$ and $\mu$ into $E$ and $B$ to make them a function of $D$ and $H$. However it is interesting to note that since translational invariance is broken, momentum is no longer conserved, which manifests itself as the divergence of $T_{\mu \nu}$ (in relativistic notation) no longer vanishing since now the derivatives act on $E$ and $B$ but also $\epsilon$ and $\mu$.

6.11

A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen. This means that all the momentum of the plane wave is absorbed by the screen.

a.

The momentum density of an electromagnetic field is given by equation (6.118) as $g = \frac{1}{c^2}(E \times H)$. Since an electromagnetic wave travels at speed $c$, the rate of momentum absorbed by the screen, per area, is $cg = \frac{1}{c}(E \times H) = \sqrt{\frac{\epsilon_0}{\mu_0}}(E \times B)$ and by Newton’s second law, this is the force per unit area, the pressure. 

For a plane wave electromagnetic wave, this gives

$$cg = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{A^2}{c} = \epsilon_0 A^2$$

Whereas the total energy density is given by (6.106) as

$$u = \frac{1}{2} \left( \epsilon_0 A^2 + \frac{1}{\mu_0 \frac{A^2}{c^2}} \right) = \epsilon_0 A^2$$

So indeed they are equal.
b.
Consider an energy flux of \( F = 1.4 \text{W/m}^2 = 1.4 \text{kg/s}^3 \) (such as that from the sun at the earth’s mean radius) and a solar sail of mass of \( 1 \text{g/m}^2 = 0.01 \text{kg/m}^2 \) of area and negligible other weight. The field energy per unit volume (and thus the pressure) is then \( F/c \) and the maximum acceleration of the sail (that is, normal incidence with no losses) will be

\[
a = \frac{1.4}{0.01(3 \times 10^8)} \frac{m}{s^2} \simeq 4.67 \times 10^{-7} \text{m/s}^2
\]  

(10)

As for the solar wind, the average pressure of the solar wind is approximately \( 1.2 \text{nPa} = 2 \times 10^{-9} \text{kg/ms}^2 \). Thus, the acceleration due to the solar wind is

\[
2 \times 10^{-9} \text{m/s}^2 = 2 \times 10^{-7} \text{m/s}^2
\]  

(11)

So, these two accelerations are comparable, about a factor of 2 different.

6.16

a.
The minimum magnetic charge of a Dirac monopole is \( g = \frac{2\pi}{e} \). The generalized Lorentz force law (from problem 6.17) is

\[
\mathbf{F} = q\mathbf{E} + q\mathbf{mB}/\mu_0 + q_e \mathbf{v} \times \mathbf{B} - q\mathbf{m}\mathbf{v} \times \epsilon_0 \mathbf{E}
\]  

(12)

Finally, the magnetic field in the median plane of a magnetic dipole (directly above the dipole) of dipole moment \( e\hbar/2m_p\hat{z} \) is.

\[
\mathbf{B}(\rho) = \frac{e\hbar \mu_0}{6m_p \rho^3} \hat{z}
\]  

(13)

So that the force on the magnetic monopole (at rest) is

\[
\mathbf{F} = \frac{\hbar^2}{3m_p \rho^3} \hat{z} = \frac{(1.055 \times 10^{-34} \text{m}^2 \text{kg/s}^{-1})^2}{3(1.673 \times 10^{-21} \text{kg}) \pi (5 \times 10^{-10} \text{m})} \rightarrow -3.529 \times 10^{-33} \text{N}
\]  

(14)

In the \( \hat{z} \) direction. Whoa, that’s tiny. Now, the work done in moving this thing in from infinity is this, integrated to infinity which gives

\[
V = 8.82 \times 10^{-24} \text{J} \simeq 10^{-5} \text{ev}
\]

Which is much less than the binding energy of an electron to a hydrogen nucleus, for example.

Also, it turns out that the median plane is actually the other plane but fortunately due to the form of the magnetic dipole field, the “real” answer is just this one, divided by \(-2\), or

\[
\mathbf{F} = 1.7645 \times 10^{-33} \text{N}
\]  

(15)

b.
If we use the binding energy above (an ok order of magnitude estimate despite the fact that it is in the wrong position), the energy of this interaction is on the order of the hyperfine splitting\(^2\) \( (4.5 \times 10^{-5} \text{ev}) \). It is also way smaller than the spin orbit splitting in a typical atom \( (0.0021 \text{ev} \text{ in sodium}) \)

6.20

Consider the dipole source (of unit strength and moment in the \( z \) direction) described by

\[
\rho(x, t) = \delta(x)\delta(y)\delta'(z)\delta(t)
\]  

(16)

\[
J_z(x, t) = -\delta(x)\delta(y)\delta(z)\delta'(t)
\]  

(17)

\(^1\)At least is has been for the last fifty years. See http://wattsupwiththat.com/2009/06/09/solar-wind-flow-pressure-another-indication-of-solar-downtrend/

\(^2\)http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/sodzee.html#c1
a.

Using equation (6.23) we see that the instantaneous coulomb potential is

\[ \Phi(x, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(x', t)}{|x - x'|} d^3x' \]

\[ = \frac{1}{4\pi \epsilon_0} \delta(t) \int \frac{\delta'(z)}{\sqrt{x^2 + y^2 + (z - z')^2}} dz' \]

\[ = -\frac{1}{4\pi \epsilon_0} \delta(t) \int \frac{\delta(z)(z - z')}{(x^2 + y^2 + (z - z')^2)^{3/2}} dz' \]

\[ = -\frac{1}{4\pi \epsilon_0} \delta(t) \frac{z}{r^3} \]

b.

Recall that the transverse and longitudinal currents are defined by the equation

\[ J = J_l + J_t, \text{ with } \nabla \times J_l = 0 \text{ and } \nabla \cdot J_t = 0 \] (23)

The easiest way to find the transverse current, it seems, will be to calculate the longitudinal current from equation (6.29) and subtract it from the total current. This gives

\[ J_l = \epsilon_0 \nabla \frac{\partial \Phi}{\partial t} = -\frac{1}{4\pi} \delta'(t) \nabla \frac{z}{r^3} \]

So that the transverse current is

\[ J_t = -\delta'(t) \left( \hat{z}\delta(x) - \frac{1}{4\pi} \nabla \frac{z}{r^3} \right) \]

\[ = -\delta'(t) \left( \hat{z}\delta(x) - \frac{1}{3} \frac{\delta(x) \hat{z}}{r^4} - \frac{1}{4\pi} \left( -3\frac{z}{r^4}, -3\frac{z}{r^4}, \frac{1}{r^3}, -3\frac{z}{r^4} \right) \right) \]

\[ = -\delta'(t) \left( \frac{2}{3} \hat{z}\delta(x) - \frac{\hat{z}}{4\pi r^3} + 3 \frac{z}{4\pi r^3} \right) \]

(25)

(26)

(27)

(28)

c.

We will proceed by first putting \( J_t \) in the form

\[ J_t = -\delta'(t) \left[ \hat{z}\delta(x) + \frac{1}{4\pi} \nabla \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \right] \]

(29)

In the coulomb gauge, the vector potential satisfies the wave equation

\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_0 J_t \]

(30)

Since the vector potential is in the form of a wave equation, which is causal, and the \( E \) and \( B \) fields follow directly from the vector potential by equation (6.31) in Jackson, the fields are indeed Causal.

We're gonna do this by fourier transforming the vector potential and transverse current in the time coordinate in order to get rid of those pesky delta functions. This gives us

\[ \nabla^2 \tilde{A} + \frac{\omega^2}{c^2} \tilde{A} = -\mu_0 \tilde{J}_t \]

(31)

\[ \tilde{J}_t = -\frac{i \omega}{2\pi} \left[ \hat{z}\delta(x) + \frac{1}{4\pi} \nabla \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \right] \]

(32)
This is the helmholtz equation with $k \equiv \omega/c$ and solution\(^3\)

\[
\tilde{A}(x, y, z, \omega) = \sum_{n=0}^{\infty} \int d^3 x' \frac{\phi_n(r)\phi_n(r')}{k^2 - k_n^2} J_l
\]  

(33)

where the $\phi$ are the solutions to the homogenous holmholtz equation

\[
\phi_n(r, \theta, \phi) = \sum_{l=0}^{\infty} (a_lj_l(nr) + b_ly_l(nr))P_l(\theta)
\]

(34)

and we have chosen only the $m = 0$ solutions since the source is azimuthally symmetric and the $j$ and $y$ are the spherical bessel functions. Now we ought to expand the current in spherical bessel functions. The delta function is easy, since the bessel functions are complete. I suppose alternatively we could note that the greens function of the helmholtz equation gives us a solution

\[
\tilde{A}(x, \omega) = \frac{\mu_0}{4\pi} \int d^3 x \tilde{J}_l \frac{e^{ik|x-x'|}}{|x-x'|}
\]

(35)

So that we can write

\[
\tilde{A}(x, \omega) = \frac{\mu_0}{4\pi} \int d^3 x \tilde{J}_l \frac{e^{ik|x-x'|}}{|x-x'|}
\]

(36)

\[
= -\frac{i\omega\mu_0}{4\pi} \int d^3 x \frac{e^{ik|x-x'|}}{|x-x'|} \left( \frac{2}{3} \hat{\delta}(x) + \frac{1}{4\pi} \nabla \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \right)
\]

(37)

\[
= -i\omega\frac{\mu_0}{4\pi} \left( \frac{e^{ikr}}{r} \frac{2}{3} \hat{\delta} + \int d^3 x \frac{e^{ik|x-x'|}}{|x-x'|} \left( \frac{1}{4\pi} \nabla \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \right) \right)
\]

(38)

\[
= -i\omega\frac{\mu_0}{4\pi} \left( \frac{e^{ikr}}{r} \frac{2}{3} \hat{\delta} - \int d^3 x e^{ik|x-x'|} \hat{\nabla}(|x-x'|) \left( \frac{1}{4\pi} \nabla \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \right) \right)
\]

(39)

\[
= -i\omega\frac{\mu_0}{4\pi} \left( \frac{e^{ikr}}{r} \frac{2}{3} \hat{\delta} - \int d^3 x e^{ik|x-x'|} \hat{\nabla}(|x-x'|) \left( \frac{1}{4\pi} \nabla \nabla \frac{1}{r} \right) \right)
\]

(40)

\[
= -i\omega\frac{\mu_0}{4\pi} \left( \frac{e^{ikr}}{r} \frac{2}{3} \hat{\delta} - \int d^3 x e^{ik|x-x'|} \hat{\nabla}(|x-x'|) \left( \frac{1}{4\pi} \nabla \nabla \frac{1}{r} \right) \right)
\]

(41)

Let's focus on the second term in integral

\[
\int d^3 x e^{ik|x-x'|} \hat{\nabla}(|x-x'|) \left( \frac{1}{4\pi} \nabla \nabla \frac{1}{r} \right) = \int d^3 x \hat{\nabla}(|x-x'|) \nabla e^{ik|x-x'|} \left( \frac{1}{4\pi} \nabla \nabla \frac{1}{r} \right)
\]

(43)

\[
= \frac{\partial}{\partial z} \hat{\nabla}(|x-x'|) e^{ik|x-x'|}|_{x'=0} + \int d^3 x \hat{\nabla}(|x-x'|) \nabla e^{ik|x-x'|} \left( \frac{1}{4\pi} \nabla \nabla \frac{1}{r} \right)
\]

(44)

(45)

Ok. I give up... for now.

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\(^3\)http://mathworld.wolfram.com/GreensFunctionHelmholtzDifferentialEquation.html