1. State the universal coefficient theorem for cohomology and use it to show that, for any space $X$, the group $H^1(X; \mathbb{Z})$ has no torsion.

Solution:

The universal coefficient theorem says that we have an exact sequence

$$0 \rightarrow \text{Ext}(H_{n-1}(X; G); \mathbb{Z}) \rightarrow H^n(X; G) \rightarrow \text{Hom}(H_n(X; \mathbb{Z}), G) \rightarrow 0.$$ 

Let $n = 1$ and $G = \mathbb{Z}$. Since $H_0(X; \mathbb{Z})$ is free abelian, the first term vanishes, so we have an isomorphism $H^1(X; \mathbb{Z}) \cong \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$. Since $\mathbb{Z}$ has no torsion, neither does $\text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$. (More generally, if $A$ and $B$ are abelian groups and $B$ has no torsion, then $\text{Hom}(A, B)$ has no torsion.)
2. Show that $\mathbb{C}P^3 \times S^8$ is not homotopy equivalent to $\mathbb{C}P^7$.

Solution:

We have $H^*(\mathbb{C}P^3 \times S^8; \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^4, \beta^2, \alpha \beta)$ with $\deg(\alpha) = 2$ and $\deg(\beta) = 8$. On the other hand, we have $H^*(\mathbb{C}P^7; \mathbb{Z}) \cong \mathbb{Z}[\gamma]/(\gamma^2)$ with $\deg(\gamma) = 2$. These two rings are isomorphic as graded abelian groups, but not as rings. For example, the former contains a nonzero element in degree 2 whose 4th power is zero, while the latter does not.
3. Compute the Poincaré series and the Euler characteristic of the flag manifold $Fl_n(\mathbb{C}^n)$.

Hint: We showed in class that the fiber bundle $\mathbb{C}P^{n-1} \hookrightarrow Fl_n(\mathbb{C}^n) \rightarrow Fl_{n-1}(\mathbb{C}^{n-1})$ satisfies the hypotheses of the Leray-Hirsch theorem.

Solution:

This was a bad hint; it should have said that the fiber bundle $\mathbb{C}P^{n-k} \hookrightarrow Fl_k(\mathbb{C}^n) \rightarrow Fl_{k-1}(\mathbb{C}^n)$ satisfies the hypotheses of the Leray-Hirsch theorem, where

$$Fl_k(\mathbb{C}^n) = \{(F_0, \ldots, F_k) \mid \dim F_i = i, F_i \subset F_{i+1}\}.$$  

(There is no natural map from $Fl_n(\mathbb{C}^n)$ to $Fl_{n-1}(\mathbb{C}^{n-1})$.) This is indeed what we showed in class.

The Leray-Hirsch theorem tells us that $H^*(Fl_k(\mathbb{C}^n); \mathbb{Z}) \cong H^*(\mathbb{C}P^{n-k}; \mathbb{Z}) \otimes H^*(Fl_{k-1}(\mathbb{C}^n); \mathbb{Z})$ as graded abelian groups. By induction, we may conclude that

$$H^*(Fl_n(\mathbb{C}^n); \mathbb{Z}) \cong H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) \otimes \cdots \otimes H^*(\mathbb{C}P^1; \mathbb{Z}),$$

and therefore that

$$p(Fl_n(\mathbb{C}^n)) = p(\mathbb{C}P^{n-1}) \times \cdots \times p(\mathbb{C}P^1) = \prod_{k=1}^{n-1} (1 + t^2 + \cdots + t^{2k}).$$

We get the Euler characteristic by plugging in $t = -1$: $\chi(Fl_n(\mathbb{C}^n)) = n!$. 

4. State the Lefschetz fixed-point theorem. Show that, for any map \( f : S^{2n} \to S^{2n} \), there exists a point \( x \in S^{2n} \) such that \( f(x) = \pm x \).

Solution:

Let \( X \) be a finite simplicial complex and \( f : X \to X \) any map, and let

\[
\tau(f) := \sum_{i \geq 0} (-1)^i \text{trace} \left( f_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q}) \right).
\]

The Lefschetz fixed-point theorem says that, if \( \tau(f) \neq 0 \), then \( f \) has a fixed point.

When \( X = S^{2n} \), we have \( \tau(f) = 1 + \deg(f) \), so \( f \) has a fixed point unless \( \deg(f) = -1 \). In this case, we have \( \deg(-f) = (-1)^{2n+1} \deg(f) = 1 \), thus \( -f \) has a fixed point.
5. Fix points \( x \in \mathbb{R}P^3 \) and \( y \in \mathbb{R}P^4 \). Show that there does not exist a map

\[ f : \mathbb{R}P^3 \times \mathbb{R}P^4 \to \mathbb{R}P^6 \]

such that the restriction of \( f \) to \( \{ x \} \times \mathbb{R}P^4 \) is the inclusion of the 4-skeleton and the restriction of \( f \) to \( \mathbb{R}P^3 \times \{ y \} \) is the inclusion of the 3-skeleton.

Hint: What would be the induced map on \( H^1(-; \mathbb{Z}_2) \)?

Solution:

We have \( H^*(\mathbb{R}P^6; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\langle \alpha^7 \rangle \), and \( H^*(\mathbb{R}P^3 \times \mathbb{R}P^4; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta, \gamma]/\langle \beta^4, \gamma^5 \rangle \). We know that \( f^*\alpha = k\beta + \ell\gamma \) for some \( k, \ell \in \mathbb{Z}_2 \). Our hypotheses about the 3 and 4-skeleta imply that \( k = \ell = 1 \). But then we have

\[
0 = f^*(0) = f^*(\alpha^7) = (\beta + \gamma)^7 = \sum_{i=0}^{7} \binom{7}{i} \beta^i \gamma^{7-i} = 35\beta^3\gamma^4 = \beta^3\gamma^4 \neq 0,
\]

which gives us a contradiction.
6. 3. Fix an integer \( n > 1 \), and let \( X \) be the complement of two points in \( \mathbb{R}P^n \). Compute (in terms of generators and relations) the ring \( H^*(X; \mathbb{Z}_2) \).

Solution:

Consider the long exact sequence for the pair \((\mathbb{R}P^n, X)\):

\[
\cdots \to H^k(\mathbb{R}P^n, X; \mathbb{Z}_2) \to H^k(\mathbb{R}P^n; \mathbb{Z}_2) \to H^k(X; \mathbb{Z}_2) \to H^{k+1}(\mathbb{R}P^n, X; \mathbb{Z}_2) \to \cdots.
\]

By excision,

\[
H^k(\mathbb{R}P^n, X; \mathbb{Z}_2) \cong H^k(D^n, D^n \setminus \{x, y\}; \mathbb{Z}_2) \cong \tilde{H}^k(S^n \vee S^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}
\]

This implies that the map \( H^k(\mathbb{R}P^n; \mathbb{Z}_2) \to H^k(X; \mathbb{Z}_2) \) is an isomorphism when \( k < n - 1 \) or \( k > n \) and an injection when \( k = n - 1 \). Analyzing the end of the sequence tells us that \( H^n(X; \mathbb{Z}_2) = 0 \) and that the cokernel of the map \( H^k(\mathbb{R}P^n; \mathbb{Z}_2) \to H^k(X; \mathbb{Z}_2) \) is isomorphic to \( \mathbb{Z}_2 \). Thus we must have

\[
H^*(X; \mathbb{Z}_2) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^n, \beta^2, \alpha \beta)
\]

with \( \deg(\alpha) = 1 \) and \( \deg(\beta) = n - 1 \).

One could produce a similar long exact sequence using Mayer-Vietoris.

One could also observe by geometric reasoning that \( X \) is homotopy equivalent to \( \mathbb{R}P^{n-1} \vee S^{n-1} \), which would give you the answer immediately.
7. Let \( X := \left( [0, 1] \times S^1 \right) / \sim \), where \((0, z) \sim (1, \bar{z})\) for each unit complex number \( z \in S^1 \). Consider the open cover \( X = A \cup B \), where
\[
A := \left( \frac{1}{5}, \frac{4}{5} \right) \times S^1 \subset X \quad \text{and} \quad B := \left( \left( [0, 2/5] \cup (3/5, 1] \right) \times S^1 \right) / \sim \subset X.
\]
Use the Mayer-Vietoris sequence for this open cover to compute \( H^*(X; \mathbb{Z}) \).

Hint: It’s not too hard to see that \( A \) and \( B \) are both homotopy equivalent to \( S^1 \), and \( A \cap B \) is homotopy equivalent to \( S^1 \sqcup S^1 \). The difficulty is in making these homotopy equivalences explicit enough so that you can really understand the maps in the long exact sequence.

Solution:

The Mayer-Vietoris sequence takes the form
\[
\cdots \to H^{k-1}(A) \oplus H^{k-1}(B) \to H^k(A \cap B) \to H^k(X) \to H^k(A) \oplus H^k(B) \to H^k(A \cap B) \to \cdots,
\]
so we need to understand the map
\[
H^k(A) \oplus H^k(B) \to H^k(A \cap B).
\]
Let’s write \( A \cap B = C \cup D \), where
\[
C = (3/5, 4/5) \times S^1 \quad \text{and} \quad D = \left( \left( [0, 1/5] \cup (4/5, 1] \right) \times S^1 \right) / \sim.
\]
Then \( A, B, C \), and \( D \) are all homotopy equivalent to \( S^1 \). We can make these identifications in such a way that the inclusions \( C \hookrightarrow A \), \( C \hookrightarrow B \), and \( D \hookrightarrow A \) are all homotopic to the identity, while the inclusion \( D \hookrightarrow B \) is homotopic to the map \( h : z \mapsto \bar{z} \). Then the map
\[
H^k(A) \oplus H^k(B) \to H^k(A \cap B) \cong H^k(C) \oplus H^k(D)
\]
is given in block form by the matrix
\[
\begin{pmatrix}
\text{id} & \text{id} \\
-\text{id} & -h^*
\end{pmatrix}.
\]
(Here the minus signs come from the funny signs that we need to make the Mayer-Vietoris sequence exact.) When \( k = 0 \), this is the matrix
\[
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}.
\]
When \( k = 1 \), this is the matrix
\[
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}.
\]
Thus we have
\[
0 \to \text{coker} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix} \to H^1(X; \mathbb{Z}) \to \ker \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \to 0,
\]
which translates to
\[
0 \to \mathbb{Z} \to H^1(X; \mathbb{Z}) \to \mathbb{Z} \to 0.
\]
This means that \( H^1(X; \mathbb{Z}) \cong \mathbb{Z} \) or \( \mathbb{Z} \oplus \mathbb{Z}_2 \). By the first problem, it must be \( \mathbb{Z} \). We also have
\[
H^2(X; \mathbb{Z}) \cong \text{coker} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \cong \mathbb{Z}_2.
\]
8. Generalizing Problem 7, let $F$ be a connected topological space and let $h : F \to F$ be a homeomorphism. Let $X := \left([0, 1] \times F\right)/\sim$, where $(0, z) \sim (1, h(z))$ for all $z \in F$. Consider the open cover $X = A \cup B$, where

$$A := (1/5, 4/5) \times F \subset X \quad \text{and} \quad B := \left(([0, 2/5] \cup (3/5, 1)] \times F\right)/\sim \subset X.$$ 

Suppose that, for all $k > 0$, the vector space $H^k(F; \mathbb{Q})$ is finite-dimensional and the linear map $h^* : H^k(F; \mathbb{Q}) \to H^k(F; \mathbb{Q})$ does not have 1 as an eigenvalue. (Note that this is the case for the map in the previous problem.) Use the Mayer-Vietoris sequence for this open cover show that

$$H^* (X; \mathbb{Q}) \cong H^* (S^1; \mathbb{Q}).$$

Solution:

As in the solution to problem 7, we need to understand the map

$$H^k(A) \oplus H^k(B) \to H^k(A \cap B) \cong H^k(C) \oplus H^k(D)$$

given in block form by the matrix

$$\begin{pmatrix} \text{id} & \text{id} \\ -\text{id} & -h^* \end{pmatrix},$$

where now the blocks have size dim $H^k(F; \mathbb{Q})$. I claim that this map is an isomorphism for all $k > 1$. Indeed, we can row reduce to get the matrix

$$\begin{pmatrix} \text{id} & \text{id} \\ 0 & \text{id} \end{pmatrix} - h^*,$$

and the fact that $h^*$ doesn’t have 1 as an eigenvalue says exactly that the determinant of this matrix is nonzero. This implies that $H^k (X; \mathbb{Q})$ vanishes for all $k > 1$, and

$$H^1 (X; \mathbb{Q}) \cong \text{coker} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cong \mathbb{Q}.$$