1. Basic TFTs

Definition. An \( n \)-dimensional TQFT is a symmetric monoidal functor 
\( Z : \text{Cob}(n) \to \text{Vect} \).

Here \( \text{Cob}(n) \) is the category of cobordisms defined by:
- Objects are closed oriented \((n - 1)\)-manifolds.
- Morphisms are cobordisms \( M \xrightarrow{B} N \).

\( B \) is an oriented \( n \)-manifold together with an identification \( \partial B \to M \sqcup N \). We consider cobordisms up to orientation-preserving diffeomorphisms.

Composition is given by gluing. The monoidal structure is the disjoint union (the empty manifold is the unit).

Therefore, \( Z(\emptyset) = \mathbb{C} \). If \( B^n \) is a closed manifold, then it is a cobordism between empty sets. The functor \( Z \) assigns an element of \( \text{End}(\mathbb{C}) \), i.e. a number.

Example: \( n = 1 \). 0-manifolds are finite collections of oriented points. We have two orientations on the point, so we get two vector spaces \( V \) and \( W \) for each.

The cap and cup diagrams give morphisms
\[
\mathbf{C} \to V \otimes W, \quad V \otimes W \to \mathbf{C}.
\]

Exercises:

1. Check that \( V^* \cong W \) and they are both finite-dimensional.
2. If \( M \) is a closed \((n - 1)\)-manifold, \( Z(\overline{M}) \cong Z(M)^* \) and \( Z(M) \) is finite-dimensional.

Now let’s consider \( n = 2 \). In \( \text{Cob}(2) \) the objects are closed 1-manifolds, i.e. finite disjoint unions of circles.

\[
Z(M) = Z(\sqcup S^1) = \otimes Z(S^1).
\]

So, \( Z(S^1) \) completely determines \( Z(M) \) for any 1-manifold.

Recall that any surface \( \Sigma \) can be obtained by gluing cups, caps and pants.
$Z$ of the pairs of pants gives maps

$$Z(S^1) \otimes Z(S^1) \to Z(S^1), \quad Z(S^1) \xrightarrow{\Delta} Z(S^1) \otimes Z(S^1),$$

which we call multiplication and comultiplication.

The disks give

$$Z(S^1) \to C, \quad C \to Z(S^1),$$

which are the trace and the unit.

So, $Z(S^1)$ has a structure of a commutative Frobenius algebra.

To summarize,

**Theorem.** There is a 1-1 correspondence between 2d TFTs and commutative Frobenius algebras.

The equivalence is given by assigning the Frobenius algebra to the circle.

### 2. Extended TFTs

Let’s consider 2-extended TFTs. We want $Z$ to assign something to $M^{n-2}$.

Let’s introduce $\text{Cob}_2(n)$:

1. Objects are closed $(n-2)$-manifolds
2. Morphisms are cobordisms between $(n-2)$-manifolds, i.e. $(n-1)$-manifolds with boundary
3. 2-morphisms are cobordisms between cobordisms, i.e. $n$-manifolds with corners.

Roughly a 2-extended TFT is a symmetric monoidal functor $Z : \text{Cob}_2(n) \to \text{some 2-category}$.

Our target 2-category is the 2-category of algebras:

- Objects are $C$-algebras
- 1-morphisms $\text{Hom}(A,B)$ are $(A,B)$-bimodules
- 2-morphisms are bimodule maps.

Note, that we can look at $\text{Hom}(\emptyset, \emptyset)$ in $\text{Cob}_2(n)$. This is precisely the category $\text{Cob}(n)$.

Similarly, $\text{Hom}_{\text{Alg}}(C,C) = \text{Vect}$. So, a 2-extended TFT gives an ordinary TFT.

**Theorem.** 2d extended TFTs are the same as (noncommutative) Frobenius algebras (finite-dimensional and semisimple).

The corresponding Frobenius algebra is $A = Z(\cdot)$.

**Proposition.** $Z(S^1)$ is the abelianization of $A$. 
Proof. By breaking the circle into two segments, we get

\[ Z(S^1) = A \otimes_{A \otimes A^{op}} A. \]

But this is the zeroth Hochschild homology, which is the same as the abelianization. \(\square\)

Corollary. For a Frobenius algebra, the center is the same as the abelianization.

Example: let \(\Gamma\) be a finite group. Define \(Z_{\Gamma}(\cdot) = C[\Gamma]\).
Then \(Z_{\Gamma}(S^1) = C[\Gamma]^\Gamma\), i.e. the space of class functions on \(\Gamma\).
Going up, we get that \(Z_{\Gamma}(\Sigma)\) is the number of points in the space (orbifold) of \(\Gamma\)-bundles on \(\Sigma\), i.e. \(#\{\text{Hom}(\pi_1(\Sigma), \Gamma)/\Gamma\}\).