The most interesting thing David heard all year: pivotal tensor categories categorify Frobenius algebras.

Main idea for today: Frobenius algebras have a really nice representation theory.

Let $A$ be a Frobenius algebra. For any $A$-module $M$ we can associate its character

$$\chi_M \in Cl(A) = \{ f : A \to C, f(ab) = f(ba) \}.$$ 

For a Frobenius algebra we can also define a certain idempotent $p_M \in Z(A)$. Since $A$ is Frobenius, $Z(A) \cong Cl(A)$ and under this isomorphism the character goes to the idempotent.

NB: tensor means monoidal, rigid.

Recall, that a Frobenius algebra $A$ is an algebra with a trace $\text{tr} : A \to C$, such that

1. $\langle a, b \rangle = \text{tr}(ab)$ is a symmetric non-degenerate bilinear form
2. $\text{tr}(ab) = \text{tr}(ba)$.

Recall: $\mathcal{C}$ tensor category is pivotal if it is equipped with natural monoidal isomorphisms $\eta_X : X^{**} \cong X$.

Examples:
- $\text{Vect}_{fd}(G)$.
- $\text{Rep}(G)$
- $\text{Rep}(H)$ for a semisimple finite-dimensional Hopf algebra
- Possibly, all fusion categories

Pivotal $\otimes$-categories have a trace $\text{tr} : \mathcal{C} \to \text{Vect}$ given by $X \mapsto \text{Hom}(1, X)$.

$$\text{Hom}(1, X \otimes Y) \cong \text{Hom}(X^*, Y) \cong \text{Hom}(1, Y \otimes X^*) \cong \text{Hom}(1, Y \otimes X).$$

This is analogous to the trace property.

We have an analog of the class functions

$$Cl(\mathcal{C}) = \{ F : \mathcal{C} \to \text{Vect}, F(X \otimes Y) \cong F(Y \otimes X), \text{ coherence} \}.$$ 

**Theorem** (Bezrukavnikov-Finkelberg-Ostrik). *We have an equivalence $Z(\mathcal{C}) \to Cl(\mathcal{C})$ given by $z \mapsto \text{Hom}(1, z \otimes -)$.***
Surjectivity is representability and injectivity is the Yoneda lemma.
In the stable $\infty$-setting we have $HH_\bullet(\mathcal{C}) = HH^\bullet(\mathcal{C})$.
Recall: $\mathcal{M}$ is a $\mathcal{C}$-module category. We have the action map $\mathcal{C} \to \text{End}(\mathcal{M})$.
We will also need

$$Z_{\text{act}} : Z(\mathcal{C}) \to \text{End}_\mathcal{C}(\mathcal{M}).$$

**Definition.** The trace $\text{tr} : \text{End}_\mathcal{C}(\mathcal{M}) \to Z(\mathcal{C})$ is the right adjoint.

**Definition.** The central idempotent associated to a module category $\mathcal{M}$ is $e_\mathcal{M} := \text{tr}(\text{id}_\mathcal{M})$.

Aside: semisimple Frobenius algebra $A$

$$A = \bigoplus X_i \text{Mat}(X_i) \to \text{End}(X_i).$$

$$e_{X_i}A = \text{Mat}(X_i).$$

**Theorem.** $e_\mathcal{M}$ is a commutative algebra in $Z(\mathcal{C})$.

Commutativity means that

$$e_\mathcal{M} \otimes e_\mathcal{M} \xrightarrow{\sigma} e_\mathcal{M} \otimes e_\mathcal{M}$$

commutes.

Recall that $\text{tr}$ was defined as a right adjoint to a monoidal functor, and so is lax monoidal.

**Theorem.** The assignment $\mathcal{M} \mapsto e_\mathcal{M}$ is a functor from the 2-category of $\mathcal{C}$-module categories with ambidextrous module functors to commutative algebras in $Z(\mathcal{C})$ with bimodules as morphisms.

Construction: $F : \mathcal{M} \to \mathcal{N}, F^R : \mathcal{N} \to \mathcal{M}$. Moreover, $F, F^R$ are module functors.

Consider $F^RF \in \text{End}_\mathcal{C}(\mathcal{M})$ and $FF^R \in \text{End}_\mathcal{C}(\mathcal{N})$. Note: $F^RF$ is a module over $1_\mathcal{M}, 1_\mathcal{M} \xrightarrow{\eta} F^RF$. Similarly for $FF^R$.

$\text{tr}(F^RF)$ is an $e_\mathcal{M}$-module $\text{tr}(\eta)$. $\text{tr}(FF^R)$ is an $e_\mathcal{N}$-module $\text{tr}(\epsilon)$.

**Proposition.** We have a natural isomorphism $\text{tr}(F^RF) \cong \text{tr}(FF^R)$. 
The LHS represents

\[ z \mapsto \text{Hom}_Z(z, \text{tr}(F^R F)) = \text{Hom}_M(Z_{\text{act}}(z), F^R F). \]

The RHS represents

\[ z \mapsto \text{Hom}_N(Z_{\text{act}}(z), FF^R). \]

**Definition.** \( \mathcal{M} \) is small if \( \text{tr} \) is conservative.

For example, \( \text{Sh}(G) \) acting on \( \text{Sh}(X) \) is small iff \( X \) is an orbit, or nilpotent thickening.

**Proposition.** For fusion categories, small is the same as indecomposable.

For example, \( \text{Vect}(G) \)-module categories are \((H, \psi \in H^2(H, \mathbb{C}^\times))\). These are given by \( \text{Vect}(G/H, \psi) \).

**Theorem (Ben-Zvi–Jordan).** The functor \( \mathcal{M} \mapsto e_\mathcal{M} \) is conservative on small modules.

**Theorem (MacLane).**

1. \( F \) is fully faithful iff the counit is an isomorphism.
2. \( F^R \) is fully faithful iff the unit is an isomorphism.
3. \( \eta, \epsilon \) are isomorphisms iff \( F \) is an equivalence.

\[ \eta : 1_{\mathcal{M}} \to F^R F. \]

\[ e_\mathcal{M} \xrightarrow{\text{tr(}\eta\text{)}} \text{tr}(F^R F) \xrightarrow{\text{tr(}\epsilon\text{)}} e_N. \]