BEILINSON-BERNSTEIN LOCALIZATION

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1. Notations

$G$ semisimple affine algebraic group, $B$ Borel subgroup, $N$ unipotent radical. $B = G/B$ is the flag variety. We also denote $\tilde{B} = G/N$.

We have an action map from $G$ to $B$. Taking the derivative, we obtain $\mathfrak{g} \to \Gamma(B, T_B)$.

We can extend it by the universal property to a map $\mu : U_{\mathfrak{g}} \to \Gamma(B, \mathcal{D}_B)$.

$\mu$ is surjective. It skernel is the ideal generated by the kernel of the trivial character $\lambda_0 : Z(\mathfrak{g}) \to \mathbb{C}$.

For example, let $G = SL_2(\mathbb{C})$. The flag variety is $B = \mathbb{P}^1 = \text{Proj} \mathbb{C}[x, y]$. $\Gamma(T_{\mathbb{P}}^1) = \Gamma(\mathcal{O}(2))$.

$\begin{align*}
e & \mapsto \frac{1}{2} x \partial_y \\
f & \mapsto \frac{1}{2} y \partial x \\
h & \mapsto \frac{1}{2} x \partial x = -\frac{1}{2} y \partial y.
\end{align*}$

The Casimir is

$$ef + fe + \frac{1}{2} h^2 = 0.$$ 

$U_{\mathfrak{g}}\lambda_0 = \Gamma(B, \mathcal{D}_B)$.

We have the following adjunction:

$$\Delta : \Gamma(B, \mathcal{D}_B) \text{-mod} \rightleftarrows \mathcal{D}_B \text{-mod} : \Gamma$$

where $\Delta(M) = M \otimes_{\Gamma(\mathcal{D}_B)} \mathcal{D}_B$.

For example, if $G = SL_2(\mathbb{C})$.

Let $M_\alpha$ be the Verma module and $L_\alpha$ its simple quotient.

Then $M_{-2}$ is mapped to the Dirac delta $\mathcal{D}$-module and $L_0$ (the trivial representation) is mapped to $\mathcal{O}_{\mathbb{P}^1}$.

**Theorem** (Beilinson-Bernstein). $(\Gamma, \Delta)$ is an equivalence of categories (i.e. $B$ is $\mathcal{D}$-affine).
Corollary. \( \Gamma \) is exact and faithful.

Proof. Step 1. Look at the associated graded level (everything is commutative).

Then

\[
\text{gr } Z(\mathfrak{g}) = \text{Sym}(\mathfrak{g})^G \xrightarrow{\lambda} \mathbb{C}.
\]

\[
\text{Sym}(\mathfrak{g}) \to \Gamma(T^*\mathcal{B}, \mathcal{O}_{T^*\mathcal{B}})
\]

is induced from the Springer resolution

\[
\mathfrak{g}^* \leftarrow T^*\mathcal{B} : \mu.
\]

Note, that \( T^*\mathcal{B} \) consists of pairs \((x,b)\), where \( b \in \mathcal{B} \) and \( x \in b \). The map is simply \( \mu(x,b) = x \). We see that \( \mu \) lands in the nilpotent cone:

\[
T^*\mathcal{B} \to \mathcal{N} \subset \mathfrak{g} \cong \mathfrak{g}^*.
\]

\[
\begin{array}{ccc}
T^*\tilde{\mathcal{B}}/H & \to & \mathfrak{h}^* \cong \mathfrak{h} \\
\downarrow & & \downarrow \\
\mathfrak{g}^* & \to & \mathfrak{h}/W
\end{array}
\]

\[
\square
\]

2. Monodromic, twisted \( cD \)-modules

Note, that we have a right \( H \)-action on \( \tilde{\mathcal{B}} \).

We have

\[
\tilde{\mathcal{B}} \xrightarrow{(\pi^*)_H} \mathcal{B}.
\]

Definition. A weakly \( H \)-equivariant \( \mathcal{D} \)-module on \( \mathcal{B} \) is a \( \mathcal{D}_\tilde{\mathcal{B}} \)-module \( M \), such that the action map \( \mathcal{D}_\tilde{\mathcal{B}} \otimes M \to M \) is \( H \)-equivariant.

We have the action map \( \alpha_M^0 : \mathfrak{h} \to \Gamma(T\tilde{\mathcal{B}}) \to \text{End}_\mathbb{C}(M) \) and the action map on the module itself \( \alpha_M^1 : \mathfrak{h} \to \text{End}_\mathbb{C}(M) \).

We define the monodromy to be \( \alpha_M = \alpha_M^1 - \alpha_M^0 \).

Definition. A weakly \( H \)-equivariant \( \mathcal{D} \)-module is monodromic if \( \alpha_M \in \mathfrak{h}^* \) is a constant.

For example, if \( \alpha_M = 0 \) we get the usual \( \mathcal{D} \)-module.

For \( \lambda : Z(\mathfrak{g}) \cong \mathfrak{h}^W \to \mathbb{C} \) a central character we can define the sheaf \( \mathcal{D}_\mathcal{B}^\lambda \) of twisted differential operators to be the quotient of the sheaf \( \mathcal{D}_\tilde{\mathcal{B}} \) by the relations that \( \mathfrak{h} \) acts by the corresponding character.
Definition (Beilinson-Bernstein). \( \Gamma, \Delta \) form an equivalence of categories \( U(g)_\lambda \text{-mod} \rightleftarrows D_\Bt \text{-mod} \).

If \( \lambda \) is regular and dominant, we have an equivalence of abelian categories. We can drop the dominance condition at the expense of going to the derived categories.

Let us also denote by \( D_H(\Bt) \) the category of monodromic \( D \)-modules without specifying the monodromy.

Theorem (Beilinson-Bernstein in families). We have an equivalence
\[
U(g) \text{-mod} \rightleftarrows D_H(\Bt) \text{-mod}.
\]

To prove it, use Barr-Beck. Want to check conservativity and continuity.

We expect conservativity to be true (follows from classical Beilinson-Bernstein).

Why does \( \gamma^* = - \otimes D_\Bt \) preserve limits? We know that \((\gamma^*, \gamma_*)\) is an adjoint pair. It turns out more is true: these are ambidextrous (two-sided) adjoints.

We have duality functors on both \( U(g) \text{-mod} \) and \( D_H(\Bt) \text{-mod} \) (Verdier duality). Both \( \gamma_* \) and \( \gamma^* \) commute with duality, so they are two-sided adjoints.

Now let us identify the monad \( T = \gamma^* \gamma_* \) (in fact, it is also a comonad). Barr-Beck says that
\[
U(g) \text{-mod} \simeq D_H(\Bt) \text{-mod}^T \simeq D_H(\Bt) \text{-mod}_T.
\]

What does it mean to be a \( T \)-algebra?

Consider the diagram
\[
\begin{array}{ccc}
D_H(\Bt) & \simeq & D_H(\Bt) \text{-mod}^T \simeq D_H(\Bt) \text{-mod}_T
\\
\downarrow & & \downarrow
\\
D_H(\Bt) & \text{and} & D_H(\Bt)
\end{array}
\]

We have the Weyl sheaf \( W \) of differential operators on \( N \setminus G/N \).
The action of \( T \) is given by the integral transform with respect to \( W \in D_H(\Bt) \text{-mod} \).

Consider the Grothendieck-Springer resolution
\[
\wt g \xrightarrow{\pi} g.
\]
This gives rise to an adjunction

\[ QC(\tilde{g}) \xrightarrow{\pi^*} QC(g). \]

Since both canonical bundles and the relative canonical bundle are trivial, both categories are Calabi-Yau and this is an ambidextrous adjunction.