1. Basics

1.1. Definitions. Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. Recall that we have an action morphism

$$O_2 \to \text{Fun}(\text{Bord}^{fr}_2, \text{Bord}^{fr}_2)$$

given by post-composing the framing with an automorphism of $\mathbb{R}^2$. Denote by

$$X = \text{Fun}(\text{Bord}^{fr}_2, \mathcal{C}) = (\mathcal{C}^{fd})^\sim$$

the space of all TFTs.

This induces a group homomorphism

$$O_2 \to \text{Aut}(X).$$

Taking based loops, we have an $E_2$ map

$$\mathbb{Z} \simeq \Omega SO_2 \to \Omega \text{Aut}(X).$$

The generator $S$ (Serre automorphism) of this $\mathbb{Z}$-action is a natural transformation from the identity functor to itself, i.e. a collection of morphisms

$$S_z : Z(\cdot) \to Z(\cdot)$$

in $\mathcal{C}$.

Explicitly, these are morphisms coming from the interval with one twist in the framing.

1.2. Framings on the circle. Consider the space of 2-framings on $S^1$. To simplify the discussion, we will only consider framings which coincide at a basepoint $0 \in S^1$.

Given any two framings $f_x, g_x : \mathbb{R} \oplus T_x S^1 \cong \mathbb{R}^2$, we obtain a map $S^1 \to GL_2$ given by $x \mapsto g_x \cdot f_x^{-1}$. Since both of them coincide at the basepoint, we get that the space of framings is a torsor over $\Omega GL_2 \cong \mathbb{Z}$.

As Noah Snyder remarked, sometimes the 2-framing induced from a 1-framing is denoted as $S^1_0$. Although the multiplication maps get complicated, other properties are simplified.\footnote{As Noah Snyder remarked, sometimes the 2-framing induced from a 1-framing is denoted as $S^1_0$. Although the multiplication maps get complicated, other properties are simplified.}
Note, that when we regard a boundary as incoming, we flip the framing on the extra \( \mathbb{R} \) summand. In particular, we get a different orientation on the circle. However, we have an orientation-reversing diffeomorphism \( \sigma : S^1 \to S^1 \) and any other orientation-reversing diffeomorphism is isotopic to it since \( \text{Diff}^+(S^1) \simeq S^1 \) is path-connected.

**Lemma.** There is an isomorphism of framed manifolds \( \partial(I \times S^1_n) \cong S^1_n \sqcup \sigma(S^1_{2-n}) \).

\( \sigma \) will be implicit from now on.

For example, just as we have a disk with an outgoing boundary \( \emptyset \to S^1_0 \), we have a disk with an incoming boundary \( S^1_2 \to \emptyset \).

Let us now understand different framings on the pairs of pants \( P \).

Let’s take a cylinder \( I \times S^1_n \) and cut out an incoming disk. That gives a pair of pants \( P_{0,n} : S^1_0 \times S^1_n \to S^1_n \).

To understand different framings on the pair of pants, recall that any two framings are related by a map \( P_{0,n} : GL_2 \). The pair of pants \( P \) retracts to a figure-eight graph \( G_8 \). The incoming circles are the two circles in the \( G_8 \) and the loop around the outgoing circle is a sum of both loops. Homotopy classes of maps \([G_8 : S^1]\) are parametrized by two winding numbers around both loops. These maps give the framings \( P_{n,m} \) on the pair of pants, which represent the morphism \( S^1_n \sqcup S^1_m \to S^1_{n+m} \).

**1.3. Structure of framed TFTs.** Let \( A \in C \) be a fully dualizable object which corresponds to a framed 2d TFT \( Z : \text{Bord}^2_2 \to C \).

For each circle we have objects \( H_n := Z(S^1_n) \in \text{End}(1) \). Pairs of pants give maps \( H_n \otimes H_m \to H_{n+m} \). In particular, \( H_n \) is a module over the algebra \( H_0 \).

\( H_0 \) is known as the Hochschild cohomology of \( A \) while \( H_1 \) is the Hochschild homology of \( A \).

We also have the trace \( H_2 \to 1 \) and the corresponding bilinear form \( H_1 \otimes H_1 \to H_2 \to 1 \).

**1.4. Oriented TFTs.** Observe, that the space of closed 2-framed 1-manifolds \( \text{Hom}_{\text{Bord}^2_2}(\emptyset, \emptyset) \) contains as a connected component the space of 2-framed circles. It has a forgetful map to the space of oriented circles, which is simply \( B \text{Diff}^+(S^1) \simeq \mathbb{CP}^\infty \). The fibers are spaces of framings on a fixed oriented circle, which is a torsor over \( LSO_2 \simeq S^1 \times \mathbb{Z} \). That means that the space of framed circles is a collection of \( S^1 \)-bundles over \( \mathbb{CP}^\infty \). Each \( S^1 \)-bundle corresponds to a different framing \( S^1_n \).

A section of one of this \( S^1 \)-bundle is an \( S^1 \)-invariant framing. The only such thing exists for the \( S^1_1 \) framing.
We get maps

$$BS^1 = \mathbb{CP}^\infty \to \text{Hom}_{\text{Bord}_{fr}}(\emptyset, \emptyset) \xrightarrow{\zeta} \text{End}_C(1).$$

That means that we have a homotopy $S^1$-action on the Hochschild homology $H_1 = ev_X \circ coev_X$. The bundles corresponding to $S^1_n$ for $n \neq 1$ are nontrivial and so do not have sections.

Homotopy fixed points for the $SO_2$ action can be identified with Calabi-Yau objects:

**Definition.** A fully dualizable object $X$ is Calabi-Yau if we are given a morphism $\eta : ev_X \circ coev_X \to 1$, which is equivariant with respect to the $S^1$-action on the Hochschild homology and is a counit of the adjunction between $ev_X$ and $coev_X$.

One has explicit forms for the adjoints

$$ev^\vee = (S \otimes \text{id}) \circ coev, \quad \vee ev = (S^{-1} \otimes \text{id}) \circ coev.$$

So, we see that in the Calabi-Yau case we have a 2-morphism between the Serre automorphism and the identity.

2. **Examples**

2.1. **Algebra-valued TFTs.** Now let $\mathcal{C} = \text{Alg}$ be the 2-category whose objects are algebras, 1-morphisms are bimodules and 2-morphisms are morphisms of bimodules. Recall the following theorem:

**Theorem.** A dualizable object $A \in \mathcal{C}$ is fully dualizable iff the evaluation map $ev_A : A \otimes A^\vee \to 1$ has both right and left adjoints.

The dual of any algebra $A$ is $A^{op}$. Indeed, the evaluation map is $A$ regarded as an $(\mathcal{C}, A \otimes A^{op})$-bimodule and the coevaluation map is $A$ regarded as an $(A^{op} \otimes A, \mathcal{C})$-bimodule. Let us denote $A^e := A \otimes A^{op}$ the enveloping algebra of $A$.

The right and left adjoints $ev^\vee$ and $\vee ev$ are $(A^e, \mathcal{C})$-bimodules together with the following natural isomorphisms

$$\text{Hom}_{A^e}(ev^\vee \otimes_M N, M) \cong \text{Hom}_M(M, A \otimes_{A^e} N)$$

$$\text{Hom}_M(A \otimes_{A^e} N, M) \cong \text{Hom}_{A^e}(N, ev^\vee \otimes_M M)$$

for any vector space $M$ and an $A^e$-module $N$. For example, taking $M = \mathcal{C}$ and $N = A^e$ we get

$$ev^\vee = \text{Hom}_C(A, \mathcal{C}), \quad \text{Hom}_{A^e}(ev^\vee, A^e) = A.$$
Moreover, since \( \text{ev}^\vee \otimes_C M \cong \text{Hom}(A, M) \), \( A \) and \( \text{ev}^\vee \) are duals. The evaluation and coevaluation maps are
\[
A \otimes \text{ev}^\vee \cong A \otimes \text{Hom}(A, C) \to C
\]
\[
C \to \text{Hom}(A, A) \cong \text{ev}^\vee \otimes_C A.
\]

In particular, that implies that \( A \) is finite-dimensional and \( \text{ev}^\vee = A^\ast \).

The adjunction for a general \( N \) follows from the tensor-Hom adjunction.

Similarly, we see that \( \text{ev} = \text{Hom}_{A^e}(A, A^e) =: A^! \) and \( A \) are duals in \( A^e - \text{mod} \). That requires \( A \) to be finitely-generated and projective as an \( A^e \)-module.

Furthermore, \( N \mapsto N \otimes_{A^e} A \) is a right adjoint, so it is left exact, i.e. \( A \) is a flat \( A^e \)-module. One can show that it implies semisimplicity. Note, that from Wedderburn’s theorem we see that every finite-dimensional semisimple algebra is finitely-generated as an \( A^e \)-module.

Conversely, if \( A \) is semisimple, \( \text{Hom} \) and \( \otimes \) are both-sided (ambidextrous) adjoints of each other and we get the required adjunction for general \( M \) and \( N \).

Therefore, we obtain the following

**Theorem.** Every object in \( \text{Alg} \) is dualizable. Fully-dualizable objects are algebras that are

- finite-dimensional
- semisimple.

One can perform similar computations for dg-algebras, we just present the result.

**Theorem.** Fully-dualizable dg-algebras are

- compact (i.e. \( \sum_i \dim H^i(A) < \infty \))
- smooth (\( A \) is compact as an \( A^e \)-module).

Note, that for dg-modules compactness is equivalent to dualizability.

2.2. **Frobenius algebras.** Using the explicit formulas for the adjoints, we see that in the case of algebras \( S \) is the \( (A, A) \)-bimodule, which is given by \( A^* = S \otimes_A A = S \). Similarly, \( S^{-1} = A^! \).

Trivialization of the Serre functor is equivalent to the data of an \( A^e \)-module isomorphism \( A \cong A^* \), i.e. an isomorphism \( A \to \text{Hom}(A, C) \). By the tensor-Hom adjunction, this is the same as a nondegenerate map \( A \otimes A \to C \). Since the original isomorphism commuted with the \( A^e \)-module structures, the trace descends to a trace on the Hochschild homology \( A \otimes_{A^e} A \to C \). Note, that since \( \text{Hom}_{\text{Alg}}(C, C) \cong \text{Vect} \) is a 1-category, the \( S^1 \)-action is trivial.
The unit $1 \in A$ is dualized to the trace map $A \to C$. In other words, $A$ is a Frobenius algebra.

**Theorem.** Homotopy fixed points of the $SO_2$ action on fully dualizable algebras are identified with Frobenius algebras.

From Wedderburn’s theorem one can see that any fully dualizable algebra has a Frobenius structure. Therefore, the natural forgetful map $\text{Hom}_G(ESO_2, (\text{Alg}^{fd})^\sim) \to (\text{Alg}^{fd})^\sim$ given by evaluating the map on the basepoint of $ESO_2$ is surjective. In fact, the $ESO_2$-action on $(\text{Alg}^{fd})^\sim$ is equivalent to the trivial action. This is not the case for other $(\infty, 2)$-categories as we will see in the next section.

2.3. **TFTs from categories.** The picture with algebras can be categorified in 2 ways.

First, there is a “horizontal” categorification, where we replace algebras thought of as categories with one object by more general categories. In this case $\mathcal{C} = \text{Cat}$ is the 2-category of linear abelian categories and Alg is a full subcategory (Eilenberg-Watts theorem).

**Theorem.** Every compactly generated category is dualizable. Fully-dualizable categories are those categories $\mathcal{D}$ possessing a Serre autoequivalence $S : \mathcal{D} \to \mathcal{D}$, such that $\text{Hom}_\mathcal{D}(M, S(N)) \cong \text{Hom}_\mathcal{D}(N, M)^*$.

Here the dual is given by the so-called restricted opposite category, i.e. the ind-completion of the opposite category of compact objects. The evaluation is given by using the Hom-pairing on the subcategory of compact objects and continuously extending it to the ind-completion.

Another way to categorify algebras is to replace algebra objects in $\text{Vect}$ by algebra objects $\text{Alg}_{St}$ in the category of stable $\infty$-categories $\text{St}$. In this case we have

**Theorem** ([BZN]). *Compactly generated categories are dualizable objects in $\text{Alg}_{St}$. Fully dualizable objects are perfectly generated algebras.*

Finally, the latter picture can be extended to a 3-categorical level. There is an $(\infty, 3)$-category $\text{MonCat}$, which roughly has the following structure:

- Objects are monoidal categories
- 1-morphisms are bimodule categories
- 2-morphisms are functors between bimodules
- 3-morphisms are natural transformations.

**Theorem** (Douglas, Schommer-Pries, Snyder). *Fusion categories are fully dualizable in $\text{MonCat}$.*
This important theorem gives a construction of the Turaev-Viro TFT as a fully extended 3d TFT.

3. Exercises

(1) Prove the formulas \( ev^\vee = (S \otimes \text{id}) \circ \text{coev} \) and \( \vee ev = (S^{-1} \otimes \text{id}) \circ \text{coev} \) by drawing the counits and units for these two adjunctions in the bordism category.

(2) Prove that in any linear category the Serre functor is unique up to a natural isomorphism if it exists. What is the Serre functor on the bounded derived category of coherent sheaves on a smooth projective scheme? When is there a Serre functor on the abelian category?

(3) What kind of structure does one need to put on an algebra in order to obtain an unoriented 2d TFT? Does every fully dualizable algebra have this structure?

References


[Te] C. Teleman, Five lectures on topological field theory, link.