Consider $\mathcal{O}_0$, the principal block of category $\mathcal{O}$ of $\mathfrak{g}$ (finite-dimensional semisimple Lie algebra).

Here category $\mathcal{O}$ is the category of finitely-generated $\mathfrak{g}$-modules, which are $\mathfrak{h}$-semisimple, $\mathfrak{b}$-locally finite.

$\mathcal{O}_0$ is the Serre subcategory of $\mathcal{O}$ that contains all simples with the same central character as the trivial representation.

$\mathcal{O}_0$ is Artinian (i.e. every object has finite length). It has enough projectives/injectives. The simples in $\mathcal{O}_0$ up to isomorphism are indexed by the Weyl group. For $w \in W$ we denote by $L_w$ the corresponding simple. $\Delta_w$ is the Verma module that has $L_w$ as its unique simple quotient. Conventions: $L_w = L(w^{-1}w_0 \cdot 0)$.

Example: $\mathfrak{g} = \mathfrak{sl}_2$:

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$  

$W = S_2$ acts by reflections along $-1$ on the line $\mathbb{C}$. Let $s$ be 0 and $e$ -2. Then we have $\Delta_s$ and $\Delta_e = L_e$ and we have a nontrivial extension

$$0 \to \Delta_e \to \Delta_s \to L_s \to 0.$$  

Toy problem: what is $[\Delta_w : L_w]$?

Define the functor

$$\mathcal{O}_0 \to \text{Rep}(\cdot \overset{f}{\underset{e}{\rightleftarrows}} \cdot, ef = 0)$$

by $M \mapsto M_0 \overset{f}{\underset{e}{\rightleftarrows}} M_{-2}$.

Exercise: show that this functor lands in the right place, and is an equivalence. Hint: look at the Casimir.

Symmetries:

1. Evident duality $D : \mathcal{O}_0 \overset{\sim}{\to} \mathcal{O}_0^{op}$, $D^2 = \text{id}$, that fixes simples.
2. $\mathcal{O}_0$ admits a graded lift $\mathcal{O}_0^Z \to \mathcal{O}_0$, where $\mathcal{O}_0^Z$ is the category of graded representations of the same quiver with $\deg e = \deg f = 1$. 

Define \( \pi_{s*} : \mathcal{O}^Z_0 \to \text{Vect}^Z \) given by

\[
(V \supset W) \mapsto W.
\]

Similarly, define \( \pi_s^* : \text{Vect}^Z \to \mathcal{O}^Z_0 \) by

\[
W \mapsto (W(1) \supset W \oplus W(2)).
\]

Observations:

- \( \pi_s^* \) is left adjoint to \( \pi_{s*} \).
- \( \pi^!_s := D\pi_s^*D = \pi^*_s(2) \) is right adjoint to \( \pi_{s*} \).
- \( D\pi_{s*}D = \pi_{s*} \).
- \( C_s = \pi_s^*\pi_{s*} \) acts on \( \mathcal{O}^Z_0 \).
- \( \pi_{s*}\pi^*_s = \text{id} \oplus \text{id}(2) \).
- \( C^2_s = C_s \oplus C_2(2) \).

In general, we have \( \mathcal{O}^Z_0, D \) and an action of \( C_s \) on \( \mathcal{O}^Z_0 \), for every simple \( s \in W \), satisfying the above relations.

Relations between the \( C_s \)'s are controlled by \( G/B \).

### 2. Geometry

\( \mathfrak{g} \)-mod with trivial central character are mapped under the Beilinson-Bernstein localization to \( D \)-mod(\( G/B \)). It has a subcategory \( \text{Hol}^r_s(G/B) \) which is identified under Riemann-Hilbert with perverse sheaves \( \text{Perv}(G/B) \).

We will denote by \( \Delta_w \) and \( L_w \) the corresponding \( D \)-modules and/or perverse sheaves.

**Definition.** The **Hecke algebra** is

\[
\mathcal{H} = D^b_m(B\backslash G/B) = D^b_m(G\backslash(G/B \times G/B)).
\]

Here \( D^b_m \) refers to the mixed perverse sheaves.

**Warning:** \( D^b_m(B\backslash G/B) \) is not the derived category of anything.

Consider a sequence

\[
D^b_m(G\backslash G/B) \xrightarrow{\text{rat}} D^b(B\backslash G/B) \xrightarrow{f_{\alpha}} D^b(G/B).
\]

We can find \( T_w, C_w \in D^b_m(G\backslash G/B) \) that map to \( \Delta_w, L_w \in D^b(G/B) \). The toy problem becomes \( [T_w : C_{w'}] =? \)

The algebra structure on \( \mathcal{H} \) is given by convolution. Note, that convolution commutes with \( D \).

\[
G/B = \bigsqcup_{w \in W} BwB/B,
\]

we denote \( j_w : BwB/B \hookrightarrow G/B \).

Let

\[
T_w = j_{w!*}BwB/B, \quad C_w = j_{w!*}BwB/B,
\]
where $BwB/B$ is the constant mixed sheaf on $BwB/B$. We have a natural map

$$j_!BwB/B \to j_*BwB/B,$$

its image is denoted as $j_*!$.

Easy geometry gives $T_wT_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ for $l$ the length function.

Moreover, $T_w(DT_{w^{-1}}) = 1 = T_e$; in fact, every object in $\mathcal{H}$ is dualizable.

If $s \in W$ is simple, there is a short exact sequence

$$0 \to 1 \to T_s \to C_s \to 0.$$

Note, that for $SL_2$ we have $BsB/B \cong \mathbb{P}^1$.

Exercises:

1. Determine the algebra structure on $K_0(\mathcal{H})$. Hint: $\{T_w\}_{w \in W}$ is a basis.

2. Characterize the basis $\{C_w\}_{w \in W}$ in terms of $\{T_w\}_{w \in W}$ and $D$.

Comments: let $X,Y$ be $G$-varieties. $\mathcal{H}$ acts on $D^b_m(B \setminus X)$.

Let $f : X \to Y$ be $G$-equivariant, then $f_*$ is an $\mathcal{H}$-module. So, we get $\mathcal{H}$-actions on (suitable) categories of coherent sheaves.