1. Introduction

Goal: understand what they are, why they are good and how we should think about them.

Ordinary category: \( \text{Hom}(X,Y) \) is a discrete set. Infinity-category: \( \text{Hom}(X,Y) \) is a topological space.

Upshot: we can use formal arguments from ordinary category theory for infinity-categories. For example, there is an analog of Barr-Beck.

2. Simplicial sets

Definition. A simplicial set is a functor \( X: \Delta^{op} \to \text{Set} \).

Here \( \Delta \) is the simplicial category:

- Objects are finite sets \([n]\).
- Morphisms are non-decreasing maps.

Denote \( X([n]) = X_n \).

Define \( d^i: [n-1] \to [n] \) for \( 0 \leq i \leq n \). It sends \( \{0, \ldots, n-1\} \) to \( \{0, \ldots, \hat{i}, \ldots, n\} \). Then we have the boundary maps

\[
X_1 \xrightarrow{d_0} X_0.
\]

Similarly, we have \( s^i: [n] \to [n-1] \) defined by \( \{0, \ldots, n\} \to \{0, \ldots, i, i, \ldots, n-1\} \).

We can define a functor \( |\cdot| : \text{SSet} \to \text{Top} \) by

\[
|X| = \sqcup_i X_i \times |\Delta^i|/ \sim,
\]

where \( |\Delta^i| \) is the standard \( i \)-simplex and we mod out by the equivalence relation given by identifying faces using the boundary maps.

Explicitly,

\[
\Delta^n = \text{Hom}_\Delta(-, [n]).
\]

There is also a simplicial set \( \Lambda^n_i \) given by removing the \( i \)-th boundary from \( \Delta^n \).

We also have a functor in the other direction \( \text{Sing} : \text{Top} \to \text{SSet} \) given by \( X \mapsto SX_n = \text{Hom}(|\Delta^n|, X) \).
These form an adjoint pair

\[ \text{SSet} \xleftrightarrow{\text{Sing}} \text{Top}. \]

Example: let \( \mathcal{C} \) be a category. There is the nerve functor \( N : \text{Cat} \rightarrow \text{SSet} \). Defined by

\[
\begin{align*}
N\mathcal{C}_0 &= \text{Ob} \mathcal{C} \\
N\mathcal{C}_1 &= \text{Hom} \mathcal{C} \\
N\mathcal{C}_2 &= \text{compositions } X \rightarrow Y \rightarrow Z.
\end{align*}
\]

3. Three Models for Infinity-Categories

Given a topological space, we want to construct a category out of it. Let objects be points \( p \in X \) and morphisms \( \text{Hom}(p, q) \) be homotopy classes of paths between \( p, q \). This is the fundamental groupoid \( \pi_{\leq 1}(X) \).

Instead of modding out by homotopies, can keep them. Let morphisms be paths, 2-morphisms be homotopies between paths etc. Proceeding in this way we get \( \pi_{\leq \infty}(X) \), the fundamental \( \infty \)-groupoid.

Consider \( \text{Sing} : \text{Top} \rightarrow \text{SSet} \). The image satisfies a special lifting property:

\[
\begin{array}{c}
\Lambda^n_i \\
\Delta^n
\end{array} \rightarrow \xrightarrow{\exists} \xrightarrow{\exists!} \begin{array}{c}
\text{Sing } X \\
\Delta^n
\end{array}
\]

for every \( 0 \leq i \leq n \).

Similarly, the nerve of a category satisfies

\[
\begin{array}{c}
\Lambda^n_i \\
\Delta^n
\end{array} \rightarrow \xrightarrow{\exists!} \begin{array}{c}
\text{NC} \\
\Delta^n
\end{array}
\]

for every \( 0 < i < n \).

Note, that if we have the lifting property \( 0 \leq i \leq n \), then we started with a groupoid. Also note that uniqueness is not good for homotopic information.

Definition. An \( \infty \)-category is a simplicial set \( K \) that satisfies
for every $0 < i < n$.

Three notions: topological categories, $\infty$-categories and simplicial categories are all equivalent.

We have a functor $\text{Cat} \rightarrow \infty$-cat. It can be generalized to the simplicial nerve functor $N_\infty$ from simplicial categories to $\infty$-categories.

\[
N_\infty(C)_0 = \text{Ob } C \\
N_\infty(C)_1 = \text{Hom } C \\
N_\infty(C)_2 = \text{maps } X \overset{f}{\to} Y \overset{g}{\to} Z \text{ together with a homotopy } h.
\]

Question: is this the only way? No, there are many.

Remarks:
(1) We can also use model categories. It is good for explicit computations, but hard to do functorial things.

Slogan: model categories to $\infty$-categories is like basis for a vector space (or local coordinates for a manifold).
(2) $\infty$-categories are stable under taking functors.
(3) Can use Dwyer-Kan simplicial localization to turn model categories into simplicial categories (or infinity-categories). The DK localization satisfies a universal property of the form

\[
\begin{array}{ccc}
\mathcal{C} & \rightarrow & \mathcal{C}[W^{-1}] \\
\downarrow & & \downarrow \\
\mathcal{D} & \rightarrow & \mathcal{D}
\end{array}
\]

If $\mathcal{C}$ is a simplicial model category, we have an equivalence $N_\Delta(\mathcal{C}_{cf}) \simeq \mathcal{C}[W^{-1}]$.

4. Limits and colimits

For a functor $F : K \rightarrow \mathcal{C}$ the limit of $F$ is defined to be the terminal object of $\text{Cone}(F)$.

\[
\text{Cone}(F) = \mathcal{C}/F = \text{SSet}_F(\Delta^n \star \mathcal{C}, \mathcal{C}).
\]
This is an $\infty$-category.

5. **Adjunctions and Barr-Beck-Lurie**

\[ \mathcal{C} \xleftrightarrow{F} \mathcal{D}. \]

They are adjoint if we are given

\[ \mathcal{D}(F(C), D) \cong \mathcal{C}(C, G(D)). \]

Define the monad $A = G \circ F : \mathcal{C} \to \mathcal{C}$.

We have $\mathcal{D} \to A\text{-mod} \to \mathcal{C}$.

**Theorem 1** (Barr-Beck-Lurie). The functor $\mathcal{D} \to A\text{-mod}$ is an equivalence if

1. $G$ is conservative
2. It preserves geometric realizations.