I offer here some background for Chapter 5 of J. J. Sakurai, *Modern Quantum Mechanics*.

1 The problem

We consider a system with a Hamiltonian $H(t)$ that changes slowly in time. We suppose that the eigenvalues of $H$ at any time $t$ are discrete and are not degenerate. Thus any time $t$, the Hamiltonian has a complete set of eigenstates with

$$H(t)|n; t\rangle = E_n(t)|n; t\rangle .$$

(1)

The phase of $|n; t\rangle$ is not determined by the eigenvalue equation, but you should think of the phase as varying only slowly with $t$.

Now suppose that at time $t = 0$ the system starts in a state $|\alpha; 0\rangle$ and evolves according to the time-dependent Schrödinger equation:

$$i\frac{d}{dt}|\alpha; t\rangle = H(t)|\alpha; t\rangle .$$

(2)

I claim that if $|\alpha; 0\rangle$ is one of the eigenstates of $H(0)$, then $|\alpha; t\rangle$ will be a phase factor times the corresponding eigenstate of $H(t)$ as long as $H(t)$ is slowly varying. Here “slowly varying” means that the time scale $\tau$ characteristic of changes in $H$ is large compared to the inverse of energy differences $E_n(t) - E_m(t)$.

2 The differential equation

Let us expand $|\alpha; t\rangle$ in the energy eigenstates:

$$|\alpha; t\rangle = \sum_n c_n(t)e^{i\theta_n(t)}|n; t\rangle ,$$

(3)
where
\[ \theta_n(t) = -\int_0^t dt' \ E_n(t') \ . \] (4)

We are interested in how the expansion coefficients \( c_n(t) \) evolve. We expect them to evolve slowly because we have put the main time dependence in the dynamical phase factor \( \exp(i\theta_n(t)) \).

Applying the Schrödinger equation, we have
\[
\sum_n E_n(t)c_n(t)e^{i\theta_n(t)}|n;t\rangle = \sum_n \left\{ i\dot{c}_n(t)e^{i\theta_n(t)}|n;t\rangle + E_n(t)c_n(t)e^{i\theta_n(t)}|n;t\rangle + c_n(t)e^{i\theta_n(t)} \frac{d}{dt}|n;t\rangle \right\} .
\] (5)

That is
\[
0 = \sum_n \left\{ \dot{c}_n(t)e^{i\theta_n(t)}|n;t\rangle + c_n(t)e^{i\theta_n(t)} \frac{d}{dt}|n;t\rangle \right\} .
\] (6)

Taking the inner product with \( \langle m;t| \) gives
\[
0 = \dot{c}_m(t)e^{i\theta_m(t)} + \sum_n c_n(t)e^{i\theta_n(t)} \langle m;t| \frac{d}{dt}|n;t\rangle .
\] (7)

Thus
\[
\dot{c}_m(t) = -c_m(t) \langle m;t| \frac{d}{dt}|m;t\rangle - \sum_{n \neq m} c_n(t)e^{i(\theta_n(t)-\theta_m(t))} \langle m;t| \frac{d}{dt}|n;t\rangle .
\] (8)

The factor multiplying \(-c_m(t)\) in the first term in Eq. (8) is purely imaginary. To see that, note that

\[
0 = \frac{d}{dt}\langle m,t|m,t\rangle
= \langle m,t| \frac{d}{dt}|m;t\rangle + \left( \frac{d}{dt} \langle m,t| \right)|m,t\rangle
= \langle m,t| \frac{d}{dt}|m;t\rangle + \left( \langle m,t| \frac{d}{dt}|m;t\rangle \right)^* .
\] (9)
This factor is of some importance, so we give it a name:

\[
\langle m, t | \frac{d}{dt} | m; t \rangle = -i \dot{\gamma}(t)
\]

where

\[
\gamma_m(t) = i \int_0^t dt' \langle m; t' | \frac{d}{dt} | m; t \rangle.
\]

The second term in Eq. (8) can be rewritten by differentiating the energy eigenvalue equation,

\[
0 = [\dot{H}(t) - E_n(t)]|n; t\rangle + [H(t) - E_n(t)] \frac{d}{dt} |n; t\rangle.
\]

Taking the inner product with \( \langle m; t \rangle \) for \( m \neq n \) gives

\[
0 = \langle m; t | \dot{H}(t)|n; t\rangle + \langle m; t|[H(t) - E_n(t)] \frac{d}{dt} |n; t\rangle,
\]

or

\[
0 = \langle m; t | \dot{H}(t)|n; t\rangle + \langle m; t|[E_m(t) - E_n(t)] \frac{d}{dt} |n; t\rangle.
\]

Thus

\[
\langle m; t | \frac{d}{dt} | n; t \rangle = \frac{\langle m; t | \dot{H}(t)|n; t\rangle}{E_n(t) - E_m(t)}.
\]

Using this result, we have

\[
\dot{c}_m(t) = i c_m(t) \dot{\gamma}(t) - \sum_{n \neq m} c_n(t)e^{i(\theta_n(t) - \theta_m(t))} \frac{\langle m; t | \dot{H}(t)|n; t\rangle}{E_n(t) - E_m(t)}.
\]

This is the exact evolution equation for \( c_m(t) \). When \( H(t) \) is slowly varying, we can drop the second term. Why? We are supposing that \( H(t) \) varies on a time scale \( \tau \) that is long compared to \( 1/(E_n - E_m) \). The second term is evidently proportional to \( 1/\tau \), so it is small. But we want to use the evolution equation to find \( c_m(t) \) after a time \( T \) over which \( H \) has changed substantially. That is, we want to find \( c_m(t) \) after a time \( T \sim \tau \). Since \( \tau/\tau = 1 \), it is not immediately evident that the second term can be neglected. However

\[
c_m(T) = \int_0^T dt \, c_m(t) + c_m(0).
\]
When we integrate the second term over $t$, the phase factor $\exp(i(\theta_n(t) - \theta_m(t)))$ oscillates inside the integral, so that the contribution from the second term is very small. The first term has no phase factor, so it has the potential to contribute to a finite change in $c_m(t)$.

Thus we have approximately

$$\dot{c}_m(t) = ic_m(t)\gamma_m(t) .$$  \hspace{1cm} (18)

The solution of this is

$$c_m(t) = e^{i\gamma_m(t)}c_m(0) .$$  \hspace{1cm} (19)

The result (19) shows that if the system starts in a particular eigenstate $N$, so that $c_N(0) = 1$ and $c_m(0) = 0$ for $m \neq N$, then as the hamiltonian slowly changes the system remains in the eigenstate $|N;t\rangle$ that evolves from the starting eigenstate. The coefficient $c_N(t)$ can, however, acquire a phase.

## 3 Berry’s phase

Let’s consider the phase $\gamma_m(t)$ in more detail. Suppose that the hamiltonian depends on several parameters $R_1, R_2, \ldots$ and that these parameters are changed over time, resulting in the slow change in the hamiltonian over time. Then the energy eigenstates also depend on $R$ and their time dependence is the result of their $R$ dependence:

$$\frac{d}{dt}|n; R(t)\rangle = \nabla_R|m; R(t)\rangle \cdot \frac{dR(t)}{dt} .$$  \hspace{1cm} (20)

Thus the phase $\gamma_m$ is

$$\gamma_m(T) = i \int_0^T dt \frac{dR(t)}{dt} \cdot \langle m; R(t)|\nabla_R|m; R(t)\rangle .$$  \hspace{1cm} (21)

This can be rewritten as an integral over the path $C$ that the parameters follow:

$$\gamma_m(T) = i \int_C dR \cdot \langle m; R|\nabla_R|m; R\rangle .$$  \hspace{1cm} (22)

Now note that the phase $\gamma_m(T)$ seems as though it should be pretty arbitrary. Suppose that I redefine the phase of $|m; R\rangle$ so that

$$|m; R\rangle \rightarrow e^{-i\lambda(R)}|m; R\rangle .$$  \hspace{1cm} (23)
Here the extra phase $\lambda(R)$ can be anything that I like. Then
\[
\nabla_R \langle m; R \rangle \rightarrow e^{-i\lambda(R)} \nabla_R \langle m; R \rangle - i (\nabla_R \lambda(R)) e^{-i\lambda(R)} \langle m; R \rangle \quad (24)
\]
and
\[
\langle m; R \nabla_R | m; R \rangle \rightarrow \langle m; R \nabla_R | m; R \rangle - i \nabla_R \lambda(R) . \quad (25)
\]
Then the phase changes by
\[
\gamma_m(T) \rightarrow \gamma_m(T) + \int_C dR \cdot \nabla_R \lambda(R) . \quad (26)
\]
That is,
\[
\gamma_m(T) \rightarrow \gamma_m(T) + \lambda(R(T)) - \lambda(R(0)) . \quad (27)
\]
We see that if we simply change the parameters from one setting to another, then the phase $\gamma_m(T)$ can be anything. However, if if we change the parameters from $R(0)$ and go along a path in the parameter space, finally coming back to the parameters we started with, then $\lambda(R(T)) - \lambda(R(0)) = 0$ and the phase is not arbitrary. The phase $\gamma_m(T)$ then depends on the geometry of the path in parameter space. It is called Barry’s phase.