1. (a) For an additive functor $F : R\text{-Mod} \to S\text{-Mod}$ and a $(R,T)$-bimodule $W$, show that $F(W)$ has a natural structure of a $(S,T)$-bimodule.

(b) $(T)$ For $(S,R)$-bimodules $V$ and $V'$ prove, that the functors $F = V \otimes_R -$ and $F' = V' \otimes_R -$ are isomorphic as additive functors if and only if $V \simeq W$ as bimodules.

(c) Prove a similar statement for functors $\text{Hom}_S(V, -)$ and $\text{Hom}_S(V', -)$.

2. $(T)$ If bimodules $P$ and $Q$ give quasi-inverse equivalences $F = \text{Hom}_R(P, -)$ and $G = \text{Hom}_S(Q, -)$ between $R\text{-Mod}$ and $S\text{-Mod}$, show that $F \simeq Q \otimes_R -$ and $G \simeq P \otimes_S -$.

[Hint. Use adjunction. For a direct computational proof, see Ex. 21.4.4]

3. (a) Prove that $V \in R\text{-Mod}$ is a progenerator iff $V^n$ is a progenerator for some $n \in \mathbb{N}$.

(b) Prove that any progenerator is a faithful module.

(c) Give an example of a non-trivial faithful projective f.g. module which is not a progenerator.

[Hint. For semisimple or commutative rings there are no such modules.]

4. $(T)$ Let $e \in R$ be an idempotent.

(a) Prove that $\text{Tr}(Re) = ReR$. (Ex. 21.4.10.)

(b) Prove that $Re$ is a progenerator of $R\text{-Mod}$ if and only if $e$ is a full idempotent, i.e. $ReR = R$, and that in this case $R$ is Morita equivalent to $eRe$. (cf. Ex. 21.4.16.)

(c) Show that if $R = \text{Mat}_n(S)$ and $e \in R$ is a full idempotent, then $R$ is Morita equivalent to $S$.

5. $(T)$ For an $R$-module $V$, show that the natural homomorphism $V^\vee \otimes_R V \to \text{End}_R(V)$ (of abelian groups) is surjective iff $V$ is f.g. projective.

[Hint. In one direction consider $\sum_{i=1}^n \phi_i \otimes v_i$ which is mapped to $I_V$. In the other direction, realize $V$ as a direct summand of $R^n$.]

6. Let $R$ be a ring all of whose f.g. projective modules are free. Prove that any ring Morita equivalent to $R$ is isomorphic to $\text{Mat}_n(R)$ for some $n$. (cf. Ex. 21.4.29.)

7. Prove that the centers of Morita equivalent rings are isomorphic without using the categorical characterization of the center.

8. Let $R$ and $S$ be Morita equivalent rings with the equivalence given by an $(R,S)$-bimodule $RP_S$.

(a) $(T)$ Prove that the lattices of left ideals in $S$ and of $R$-submodules in $RP$ are isomorphic.

(b) $(T)$ Prove that the lattices of (two-sided) ideals in $R$ and $S$ are isomorphic. (Ex. 21.4.13.)

[Hint. Show that both lattices are isomorphic to the lattice of $(R,S)$-submodules of $RPs$.]

(c) Prove that the above isomorphism of lattices preserves products of ideals.

(d) Find the number of left ideals in $\text{Mat}_3(\mathbb{F}_7)$. 