Worksheet 8

Math 202

Monday, May 20, 2024

Say hello to your group members, and learn their names if you haven't already. What topic did they choose for their final project?

This worksheet is just to discuss in class, not to write up, since you'll be busy working on your projects.

Continue to approximate the earth as a sphere of radius $R = 4000$ miles or 6400 kilometers.

We've analyzed a number of map projections, and looked at pictures of many more: some preserved areas, some preserved angles, some preserved distances in one direction but not other directions... This leads us to wonder whether it's even possible to make a perfect map of the earth, or just part of the earth, that preserves all areas, angles, and distances.

Consider a circle of radius r miles or kilometers on the surface of the earth: by this I mean that you choose some point to be the center, and take all the points whose distance from that center, measured along the surface of the earth, is r. In most of our map projections, that circle on the earth would not appear as a circle on the map. But do you agree that on a projection that preserved all distances, it would appear as a circle?

Take that circle of radius r on the earth and find its circumference in terms of r and R. Your answer should be a bit less than $2\pi r$; is this clear from the formula? Is it clear geometrically?

If someone in your group has studied Taylor series, they might remember that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Use this to expand your formula for the circumference, and make any simplifications that jump out at you. Notice the the leading term, the fact that the second term is negative, and that the later terms matter very little when r is small compared to R.

Use `desmos.com` or another piece of software to graph your function and the function $2\pi r$. You might have to zoom way out to see any difference. At some point your function starts to decrease, and even becomes negative; is that meaningful in terms of the question you're trying to answer? For fun, also graph the first two terms of your Taylor series expansion.

On a flat piece of paper, a circle of radius r has a circumference of $2\pi r$. Does this convince you that no map can preserve all distances? Discuss it with your group. Think of different ways of explaining this argument that might be clearer or more convincing.

I claim that if a map distorts some distances then it must distort some angles, or some areas, or both. Does this make sense? Discuss this with your group. It might help to think about little squares on the surface of the Earth again. It follows that if a map preserves all angles and areas, then it must preserve all distances; does this make sense? Conclude that no map preserves both angles and areas.

Since there can be no perfect map, discuss what balance of distortions would make a "pretty good" map in your view. Consider maps of the whole world but also maps of countries or continents, which might call for different choices. It might be fun to refer again the catalogue on `map-projections.net`.

If you have time to spare, go back to your circle of radius r on the earth and find its area, as follows. Put the center of the circle at the north pole. Use the Lambert cylindrical projection, which you know preserves areas; then your circle around the the north pole corresponds to a wide rectangle at the top of the map. Find the area of that rectangle as a function of r and R. Use a computer to graph your function; it should be approximately πr^2 when r is small, and significantly less than πr^2 when r is big. If you want the Taylor series for cosine, it's

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$