Worksheet 11

Monday, November 18, 2019

Math 205

1. You have been assigned into groups of three for the week. Write down your colleagues' names and email addresses.

2. Let X and Y be sets. A function $f\colon X \to Y$ is called *injective* or *1-to-1* if it takes distinct elements of X to distinct elements of Y: that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

   It is called *surjective* or *onto* if every element of Y gets hit by some element of X: that is, for every $y \in Y$ there is an $x \in X$ with $f(x) = y$.

   It is called *bijective* if it's both injective and surjective.

   Think of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is injective but not surjective. (It doesn't have to be a nice-looking function, it could be piecewise or whatever.) Think of one that's surjective but not injective. Think of one that's bijective but not just the identity function $f(x) = x$.

3. We want to compare the sizes of two sets X and Y. Our notation for the *size* or *cardinality* of a set X will be $|X|$. We say that $|X| \leq |Y|$ if there is an injection $f\colon X \to Y$. We say that $|X| \geq |Y|$ if there is a surjection $f\colon X \to Y$. We say that $|X| = |Y|$ if there is a bijection $f\colon X \to Y$.

   The notation suggests that if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$, but this is not obvious from the definition. I'll explain the proof next time, but for now let's take it on faith.

   Is it obvious that if $|X| \leq |Y|$ then $|Y| \geq |X|$?

   We say that a set is countable if $|X| = |\mathbb{N}|$, where $\mathbb{N}$ is the natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

   Does this agree with our definition from last week?

4. Let I be the open interval $(0,1)$, that is,

$$I = \{x \in \mathbb{R} : 0 < x < 1\}.$$

Last week we saw that there is no surjective function $\mathbb{N} \to I$. (Do you agree?) So I and $\mathbb{N}$ are not the same size.

Argue that $|I| \leq |\mathbb{R}|$ and, more surprisingly, $|\mathbb{R}| \leq |I|$. That is, cook up an injection $f\colon I \to \mathbb{R}$ and $g\colon \mathbb{R} \to I$.

5. Let I^2 denote the set of pairs (x, y), where $x \in I$ and $y \in I$. Draw a picture of I^2.

Consider the function $f\colon I^2 \to I$ given by

$$f(0.a_0a_1a_2a_3\ldots,\ 0.b_0b_1b_2b_3\ldots) = 0.a_0b_0a_1b_1a_2b_2\ldots$$

Is this injective? Surjective?

Conclude that $|I| = |I^2|$.

Argue that $|\mathbb{R}| = |\mathbb{R}^2|$, where $\mathbb{R}^2$ is the plane.

6. For any set X, the *power set* $P(X)$ is the set of all subsets of X.

   Consider the function $f\colon P(\mathbb{N}) \to I$ that sends a subset $A \subset N$ to a decimal of the form $0.0010010111101\ldots$, where there's a 1 in the n^{th} place if $n \in A$, and a 0 if not. Is this function injective? Surjective?

   Consider the function $g\colon I \to P(\mathbb{N})$ that takes a number $x \in I$, writes it in *binary* as $0.a_0a_1a_2\ldots$, and sends it to the set

   $$A = \{n \in N : a_n = 1\}$$

   of the natural numbers. Is this function injective? Surjective?

   Conclude that $|P(\mathbb{N})| = |I|$, and that $|P(\mathbb{N})| = |\mathbb{R}|$.