

3. The paradox of Gabriel's horn, or Toricelli's trumpet.

- (a) Sketch the region in the first quadrant that lies to the right of the line $x = 1$ and below the curve $y = 1/x$. Argue that its area is infinite, in two ways: first, do an integral; second, draw some boxes of area $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc. that fit inside it, and argue that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

by grouping terms as we did on Wednesday. (If you were gone on Wednesday, ask one of your group members to explain this.)

- (b) Sketch the region in the first quadrant that lies to the right of the line $x = 1$ and below the curve $y = 1/x^2$. Argue that its area is finite, in two ways: first, do an integral; second, draw some boxes of area $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$, etc. that fit inside, and argue that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots < 2$$

by grouping terms as we did on Wednesday.

- (c) Sketch the 3-dimensional "horn" obtained by revolving the first region around the x -axis. Argue that its volume is finite, by slicing it into thin disks and doing an integral involving $1/x^2$.

But its cross-section has infinite area. How is this possible?

Sometimes this paradox is phrased in terms of painting the outside of the horn, whose area is also infinite, or filling it with paint.

4. Here's another paradox to critique. Among the positive integers, some are interesting – for example, 7 – and some are not interesting – for example, 81792865203981. So take the set of uninteresting numbers, and let n be its smallest element. But the first uninteresting number sounds pretty interesting, doesn't it! So we get a contradiction, and the set of uninteresting numbers must have been empty – it must be that every number is interesting.