

Worksheet 9

Monday, November 11, 2019

Math 205

1. You have been assigned into groups of three for the week. Write down your colleagues' names and email addresses.

2. A set X is called *countable* if you can put its elements in an infinite list like this:

$$X = \{x_0, x_1, x_2, x_3, \dots\}.$$

The natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ are obviously countable. To warm up, argue that the integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

are countable.

3. Argue that the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

are countable.

4. Argue that for any list of real numbers

$$x_0, x_1, x_2, x_3, \dots$$

with each x_i between 0 and 1, you can cook up another real number y between 0 and 1 that is not on the list.

5. Conclude that the set of real numbers is *uncountable*, and indeed for any real numbers $a < b$, the set of real numbers between a and b is uncountable. (Is this obvious from #4? Or does it require some explanation?)

6. Last week we saw that between any two real numbers $a < b$ there is at least one rational number, and at least one irrational number. So maybe you could hope to pair them off. But this week we've seen that there are uncountably many irrational numbers in that interval, and only countably many rational numbers. Is this a paradox, or is it ok?

7. (Optional, if you have time to kill at the end of the hour.)

A real or complex number x is called *algebraic* if it is a root of a polynomial with rational coefficients: that is, if

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0$$

for some rational numbers a_0, \dots, a_n . Otherwise x is called *transcendental*.

For example, $\sqrt{2}$ is irrational, but it is algebraic because it is a root of $x^2 - 2$. And $\sqrt{3} + \sqrt[3]{2}$ is algebraic because it is a root of $x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$. But there are more algebraic numbers than can be expressed using n^{th} roots like this: for example, the roots of $x^5 - 10x + 2$ cannot, as you'll learn in Math 446.

On the other hand, e and π are transcendental.

Argue that the set of algebraic numbers is countable. Thus the set of transcendental numbers is uncountable.