1. You have been assigned into groups of three for the week. Write down your colleagues’
names and email addresses.

2. Let $X$ and $Y$ be sets. A function $f : X \to Y$ is called *injective* or *1-to-1* if it takes
distinct elements of $X$ to distinct elements of $Y$: that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

It is called *surjective* or *onto* if every element of $Y$ gets hit by some element of $X$: that
is, for every $y \in Y$ there is an $x \in X$ with $f(x) = y$.

It is called *bijective* if it’s both injective and surjective.

Think of a function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective. (It doesn’t have
to be a nice-looking function, it could be piecewise or whatever.) Think of one that’s
surjective but not injective. Think of one that’s bijective but not just the identity
function $f(x) = x$. 

3. We want to compare the sizes of two sets $X$ and $Y$. Our notation for the size or cardinality of a set $X$ will be $|X|$. We say that $|X| \leq |Y|$ if there is an injection $f: X \to Y$. We say that $|X| \geq |Y|$ if there is a surjection $f: X \to Y$. We say that $|X| = |Y|$ if there is a bijection $f: X \to Y$.

The notation suggests that if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$, but this is not obvious from the definition. I’ll explain the proof next time, but for now let’s take it on faith.

Is it obvious that if $|X| \leq |Y|$ then $|Y| \geq |X|$?

We say that a set is countable if $|X| = |\mathbb{N}|$, where $\mathbb{N}$ is the natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots \}.$$ 

Does this agree with our definition from last week?
4. Let $I$ be the open interval $(0, 1)$, that is,

$$I = \{x \in \mathbb{R} : 0 < x < 1\}.$$ 

Last week we saw that there is no surjective function $\mathbb{N} \to I$. (Do you agree?) So $I$ and $\mathbb{N}$ are not the same size.

Argue that $|I| \leq |\mathbb{R}|$ and, more surprisingly, $|\mathbb{R}| \leq |I|$. That is, cook up an injection $f : I \to \mathbb{R}$ and $g : \mathbb{R} \to I$. 
5. Let $I^2$ denote the set of pairs $(x, y)$, where $x \in I$ and $y \in I$. Draw a picture of $I^2$.

Consider the function $f: I^2 \to I$ given by

$$f(0.a_0a_1a_2a_3\ldots, 0.b_0b_1b_2b_3\ldots) = 0.a_0b_0a_1b_1a_2b_2\ldots$$

Is this injective? Surjective?

Conclude that $|I| = |I^2|$.

Argue that $|\mathbb{R}| = |\mathbb{R}^2|$, where $\mathbb{R}^2$ is the plane.
6. For any set $X$, the power set $P(X)$ is the set of all subsets of $X$.

   Consider the function $f : P(\mathbb{N}) \to I$ that sends a subset $A \subseteq N$ to a decimal of the form $0.0010010111101\ldots$, where there’s a $1$ in the $n^{\text{th}}$ place if $n \in A$, and a $0$ if not. Is this function injective? Surjective?

Consider the function $g : I \to P(\mathbb{N})$ that takes a number $x \in I$, writes it in binary as $0.a_0a_1a_2\ldots$, and sends it to the set

$$A = \{n \in N : a_n = 1\}$$

of the natural numbers. Is this function injective? Surjective?

Conclude that $|P(\mathbb{N})| = |I|$, and that $|P(\mathbb{N})| = |\mathbb{R}|$. 