Worksheet 3

Math 206

Monday, October 9, 2023

Write down your group members' names and email addresses. Do they have any plans to watch the annular eclipse next weekend?

Two weeks ago you found many ways of understanding the formula

$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}.$$

Last week you found a formula for $1^2 + 2^2 + \cdots + n^2$ using a little calculus, a lot of algebra, and the formula for $1 + 2 + \cdots + n$. If you had time you might have found a formula for $1^3 + 2^3 + \cdots + n^3$ or even $1^4 + 2^4 + \cdots + n^4$ by the same method.

Suppose you didn't know last week's formula for $1^2 + 2^2 + \cdots + n^2$, but you suspected it might be a polynomial of degree 3 in n: that is,

$$1^2 + 2^2 + \cdots + n^2 = an^3 + bn^2 + cn + d$$

for some numbers a, b, c, and d. You could plug in four different values for n to get four equations involving a, b, c, and d, and then you could try to solve them...

Warm up by thinking about

$$1 + 2 + \cdots + n = an^2 + bn + c.$$

Plug in $n = 1$ to get one equation involving a, b, and c, then $n = 2$ to get another, then $n = 3$ to get a third, and then solve. Does your answer agree with the formula from two weeks ago?

It would have been easier to solve if you'd plugged in $n = 0$, 1, and 2 rather than 1, 2, and 3. But how do we make sense of $1 + 2 + \cdots + n$ when $n = 0$? Discuss it until you have an explanation that everyone in the group is happy with. Then go ahead and solve this simpler system of equations.

Now do the sum of squares: assume there's a formula of the form

$$1^2 + 2^2 + \cdots + n^2 = an^3 + bn^2 + cn + d,$$

plug in $n = 0$, 1, 2, and 3, and solve the resulting system of equations to find a, b, c, and d. Try to keep organized notes about how you solved it – don't just cram everything into the margin of this worksheet. Do you get the same answer as last week? Would you say this is easier or harder than last week? More or less illuminating?

Next you could think about

$$1^3 + 2^3 + \cdots + n^3 = an^4 + bn^3 + cn^2 + dn + e,$$

but the system of five equations is going to be painful to solve by hand. If someone in your group has a computer, search for "WIMS linear solver" and type the equations in there, or use another piece of software if you prefer. Keep going with

$$1^4 + 2^4 + \cdots + n^4 = an^5 + bn^4 + cn^3 + dn^2 + en + f$$

and so on.

Notice that you haven't really proved that your formulas for $1^2 + 2^2 + \cdots + n^2$ and $1^3 + 2^3 + \cdots + n^3$ and so on are correct for all n: you've only proved that *if* there's a formula of the kind you were looking for, then it has to be the one you found. How could you prove that the formulas you found are really correct? Can you think of more than one way?