Worksheet 4

Math 206

Monday, October 16, 2023

Write down your group members' names and email addresses. Did they manage to see the annular eclipse through the clouds on Saturday?

Every week I split our class of 20 into groups of 3 or 4 students. How many ways are there to choose a group of 3 students out of a class of 20? (Discuss it until everyone in your group is satisfied with the reasoning.) How many ways are there to choose a group of 3 people that includes you? How many ways are there to choose a group of 3 people that *doesn't* include you? Do your last two answers add up to equal your first answer? Do you see why they should?

The number of ways to choose k things out of a set of n things is denoted $\binom{n}{k}$, pronounced "n choose k." We have a formula

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1}. \tag{1}$$

Does this make sense to you? Make a triangle where on the n^{th} line you list $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, up through $n = 7$. Be sure to include $n = 0$, even though it's a little weird to think about choosing zero things from a set of zero things. Have you encountered this triangle before?

Do you know any other formulas for $\binom{n}{k}$? If so, how are they related to formula (1) above?

I claim that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{2}$$

Hopefully you saw this earlier with $n = 20$ and $k = 3$. Does it make sense in general, using the definition of $\binom{n}{k}$ as the number of ways to choose k things out of a set of n things? (Discuss it until everyone in your group is satisfied with the reasoning. If someone isn't convinced or doesn't know what you're talking about, keep discussing it!) Can you see it in the triangle you made earlier? Can you obtain formula (2) algebraically from formula (1)?

I claim that

$$\binom{n}{k} = \binom{n}{n-k}. \tag{3}$$

Does this make sense using the definition in terms of the number of ways to choose k things from a set of n things? Can you see it in the triangle you made earlier? Can you obtain formula (3) algebraically from formula (1)?

Multiply out $(x+1)^2$, then $(x+1)^3$, then $(x+1)^4$. Can you articulate how each step is related to going from one row to the next in the triangle you made earlier? Can you see what $(x+1)^7$ will be without tediously multiplying out $(x+1)^5$ and $(x+1)^6$ first? Can you give a satisfying explanation for why the coefficient of x^k equals the number of ways of choosing k things out a set of 7 things?

What about $(x-1)^3$ and $(x-1)^4$ and so on, which came up when you were doing worksheet 2?

What about $(x+y)^3$ and $(x+y)^4$ and so on? The general statement about $(x+y)^n$ is called the *binomial theorem*, and the numbers $\binom{n}{k}$ are often called the *binomial coefficients*.

If you have time to kill, think about plugging $n = 1/2$ into formula (1), for $k = 0, 1, 2, \ldots$ how far would it be reasonable to go? What would the binomial theorem have to say about $(1+x)^{1/2}$? Does it give sensible answers for $\sqrt{2}$ and $\sqrt{3}$?

What about $(1+x)^{-1}$, or $(1-x)^{-1}$?

If you still have time to kill, think about what a *trinomial theorem* might look like: what can you say about the expansion of $(x+y+z)^7$.