Worksheet 5

Math 206

Monday, October 23, 2023

Write down your group members' names and email addresses. Do they have any pets?

Given a function f that takes integers as input and returns integers, or rational numbers, or real numbers or whatever as output, let's define

$$\nabla f(n) = f(n) - f(n-1).$$

That's an upside-down delta, for "difference." The TEX command is `\nabla`.

If $f(n) = n^2$, find ∇f. Do the same for $f(n) = n^3$.

Convince yourselves that if you have two functions f and g with $f(0) = g(0)$ and $\nabla f(n) = \nabla g(n)$ for all n, then $f(n) = g(n)$ for all n.

Last week we studied the binomial coefficients

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1}$$

and saw that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Can we use this to find $\nabla f(n)$ if $f(n) = \binom{n}{3}$? What if $f(n) = \binom{n}{4}$?

Going the other way, suppose we want to find a function f with

$$\nabla f(n) = 3\binom{n}{4} + 9\binom{n}{3} + 10\binom{n}{2} + 5\binom{n}{1} + \binom{n}{0}.$$

How can we do it without working hard? What if we also require that $f(0) = 7$?

Let

$$f(n) = 0 + 1 + 2 + \cdots + n,$$

and let

$$g(n) = \binom{n+1}{2}.$$

Convince yourselves that $f(0) = g(0)$, and $\nabla f = \nabla g$, so $f = g$.

We've seen that $\binom{n}{0} = 1$ for all n, and $\binom{n}{1} = n$. Expand out $\binom{n}{2}$ as a polynomial in n. Can you write n^2 in terms of $\binom{n}{2}$ and $\binom{n}{1}$ and $\binom{n}{0}$?

Let

$$f(n) = 0^2 + 1^2 + \cdots + n^2.$$

Find $\nabla f(n)$. Can you find a function $g(n)$, built from binomial coefficients, with $\nabla g = \nabla f$ and $g(0) = f(0)$? Does it agree with what you've found in past weeks?

Keep going with

$$f(n) = 0^3 + 1^3 + \cdots + n^3$$

and

$$f(n) = 0^4 + 1^4 + \cdots + n^4$$

and so on.

If you have time to kill, consider the following. If f is a polynomial of degree d, then ∇f is a polynomial of degree $d - 1$, and $\nabla^2 f = \nabla\nabla f$ is a polynomial of degree $d - 2$, and so on, until

$$\nabla^d f = \overbrace{\nabla\nabla\cdots\nabla}^{d \text{ times}} f$$

is a constant function. In the write-ups, some of you have been using the converse: if $\nabla^d f$ is a constant function, then f is a polynomial of degree $\leq d$. Can you use the ideas we've explored this week to explain why this is true?