Worksheet 6

Math 206

Monday, October 30, 2023

We need some compact notation for $1^2 + 2^2 + \cdots + n^2$ and $1^3 + 2^3 + \cdots + n^3$ and so on, so let's define

$$f_p(n) = 1^p + 2^p + \cdots + n^p.$$

Depending on how long you persevered on worksheet 3, you might have found:

$$\begin{aligned}
f_1(n) &= \tfrac{1}{2}n^2 + \tfrac{1}{2}n \\
f_2(n) &= \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n \\
f_3(n) &= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 \\
f_4(n) &= \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n \\
f_5(n) &= \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2 \\
f_6(n) &= \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n \\
f_7(n) &= \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2 \\
f_8(n) &= \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n \\
f_9(n) &= \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{3}{20}n^2 \\
f_{10}(n) &= \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - n^7 + n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n \\
f_{11}(n) &= \tfrac{1}{12}n^{12} + \tfrac{1}{2}n^{11} + \tfrac{11}{12}n^{10} - \tfrac{11}{8}n^8 + \tfrac{11}{6}n^6 - \tfrac{11}{8}n^4 + \tfrac{5}{12}n^2 \\
f_{12}(n) &= \tfrac{1}{13}n^{13} + \tfrac{1}{2}n^{12} + n^{11} - \tfrac{11}{6}n^9 + \tfrac{22}{7}n^7 - \tfrac{33}{10}n^5 + \tfrac{5}{3}n^3 - \tfrac{691}{2\,730}n \\
f_{13}(n) &= \tfrac{1}{14}n^{14} + \tfrac{1}{2}n^{13} + \tfrac{13}{12}n^{12} - \tfrac{143}{60}n^{10} + \tfrac{143}{28}n^8 - \tfrac{143}{20}n^6 + \tfrac{65}{12}n^4 - \tfrac{691}{420}n^2 \\
f_{14}(n) &= \tfrac{1}{15}n^{15} + \tfrac{1}{2}n^{14} + \tfrac{7}{6}n^{13} - \tfrac{91}{30}n^{11} + \tfrac{143}{18}n^9 - \tfrac{143}{10}n^7 + \tfrac{91}{6}n^5 - \tfrac{691}{90}n^3 + \tfrac{7}{6}n
\end{aligned}$$

Two patterns jump out right away: the leading term of $f_p(n)$ is always $\frac{1}{p+1}n^{p+1}$, and the second term is always $\frac{1}{2}n^p$. How many more patterns can you find?

Use a computer to factor some of these. Now what patterns do you see? Can you explain any of them?