# Worksheet 7 (revised)

Math 206

Monday, November 6, 2023

Last week, by staring deeply into the formulas for $1^p + 2^p + \cdots + n^p$ for $p$ from 1 up to 14, we discovered Faulhaber's formula:

$$\frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \frac{p}{12}n^{p-1} - \frac{1}{120}\binom{p}{3}n^{p-3} + \frac{1}{252}\binom{p}{5}n^{p-5}$$
$$- \frac{1}{240}\binom{p}{7}n^{p-7} + \frac{1}{132}\binom{p}{9}n^{p-9} - \frac{691}{32760}\binom{p}{11}n^{p-11} + \frac{1}{12}\binom{p}{13}n^{p-13} - \cdots$$
$$\pm \text{ stop at } n \text{ or } n^2.$$

But the numbers $\frac{1}{12}$, $\frac{1}{120}$, $\frac{1}{252}$, and so on seemed to come out of nowhere, especially $\frac{691}{32760}$.

We won't get a deeply satisfying explanation for these numbers, but over the next two weeks we'll organize them into something called a generating function, which will make them feel a little less random.

To get a feeling for generating functions, let's start with the expression

$$y = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

We're just going to manipulate this formally, not worry about questions of convergence or try to evaluate it at particular values of $x$. Convince yourselves that

$$y = 1 + x \cdot y.$$

Solve for $y$ to see that (at least in a formal sense) we have $y = \frac{1}{1-x}$.

Next consider

$$z = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots . \qquad (*)$$

Convince yourselves that

$$z - xz = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

Re-write the right-hand side as $\frac{1}{1-x}$ and solve for $z$.

To understand (∗) another way, take

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

and square both sides. Squaring the right-hand side will require some careful bookkeeping.

To understand (∗) a third way, take

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

and take the derivative of both sides.

Next, find similar expression for $\frac{1}{(1-x)^3}$. You could either multiply your expressions for $\frac{1}{1-x}$ and $\frac{1}{(1-x)^2}$, or you could take the derivative of $\frac{1}{(1-x)^2}$. Do you notice a connection to Pascal's triangle? What do you guess will happen with $\frac{1}{(1-x)^4}$?

Given any sequence of numbers, we can make a formal power series called a *generating function* that has those numbers as coefficients, and we can hope to write it in a nice compact form. For example, we could take the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \ldots,$$

which are defined by the fact that one is the sum of the previous two, and organize them into a generating function:

$$w = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots$$

Convince yourselves that $w = 1 + xw + x^2w$. Solve for $w$.

Knowing that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, we can write

$$\frac{1}{1-x-x^2} = \frac{1}{1-(x+x^2)} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots .$$

On the right-hand side, use the binomial theorem to exand $(x+x^2)^2$ and $(x+x^2)^3$ and so on – maybe go up through $(x+x^2)^5$. This gives a connection between the Fibonacci numbers and Pascal's triangle; what is it?

If you've taken math 253, you've seen that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots .$$

If you haven't seen this before, you can at least convince yourself that both sides satify $f'(x) = f(x)$ and $f(0) = 1$.

Building on this formula for $e^x$, convince yourselves that

$$1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots = \frac{e^x - 1}{x}.$$

In principle then, we can find similar expression for $\frac{x}{e^x-1}$ by writing

$$\frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots} = \frac{1}{1 + (\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots)},$$

and expanding that as

$$1 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right) + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right)^2 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right)^3 + \cdots$$

Multiply this out till you get tired, then ask Wolfram Alpha for "power series for x/(e^x-1) to order 20" and check your work. Notice that the coefficient of $x^{12}$ involves 691, which suggests a connection to Faulhaber's formula. We'll dig deeper into this next week.