Worksheet 8 (typos corrected)

Math 206

Monday, November 13, 2023

From our investigations two weeks ago, we think there are numbers $a_1, a_2, a_3, \ldots$ such that for all positive integers n and p, the sum $1^p + 2^p + \cdots + n^p$ equals

$$\frac{n^{p+1}}{p+1} + a_1 n^p + a_2 p n^{p-1} + a_3 \binom{p}{2} n^{p-2} + a_4 \binom{p}{3} n^{p-3} + \cdots + a_p \binom{p}{p-1} n^1.$$

For example, we found that $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{12}$, $a_3 = 0$, $a_4 = -\frac{1}{120}$, and we found quite a few more.

Plugging in $n = 1$, we get

$$1 = \frac{1}{p+1} + a_1 + a_2 p + a_3 \binom{p}{2} + a_4 \binom{p}{3} + \cdots + a_p \binom{p}{p-1}.$$

(Where did the 1 on the left-hand side come from?)

Plugging in different values of p, we get

$$\begin{array}{llcccccccccccc}
(p=1) & \quad 1 & = & \frac{1}{2} & + & a_1 \\
(p=2) & \quad 1 & = & \frac{1}{3} & + & a_1 & + & 2a_2 \\
(p=3) & \quad 1 & = & \frac{1}{4} & + & a_1 & + & 3a_2 & + & \frac{3!}{2! \cdot 1!} a_3 \\
(p=4) & \quad 1 & = & \frac{1}{5} & + & a_1 & + & 4a_2 & + & \frac{4!}{2! \cdot 2!} a_3 & + & \frac{4!}{3! \cdot 1!} a_4 \\
(p=5) & \quad 1 & = & \frac{1}{6} & + & a_1 & + & 5a_2 & + & \frac{5!}{2! \cdot 3!} a_3 & + & \frac{5!}{3! \cdot 2!} a_4 & + & \frac{5!}{4! \cdot 1!} a_5 \\
 & & \vdots
\end{array}$$

(Does it make sense how these came from the thing above?)

We can solve these one at a time to get $a_1 = \frac{1}{2}$, then $a_2 = \frac{1}{12}$, then $a_3 = 0$, then $a_4 = -\frac{1}{120}$, and we could keep going as long as we want.

But last week we saw that if we have a sequence of numbers, we can use them as the coefficients of a "generating function" and hope to manipulate it into a simpler form. To do that here requires some care...

In the first line above, multiply through by x^2. In the second line, multiply through by x^3 and divide through by 2!. In the third line, multiply through by x^4 and divide through by 3!, and keep going.

Adding up all the left-hand sides, we get $x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$. Convince yourselves that this is $x(e^x - 1)$.

The first column on the right-hand side adds up to $\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, which is $e^x - x - 1$.

Convince yourselves that the second column adds up to $a_1 x(e^x - 1)$.

And the third column adds up to $a_2 x^2(e^x - 1)$.

And the fourth column adds up to $a_3 \frac{x^3}{2!}(e^x - 1)$.

Keep going with the fifth and sixth columns.

Putting it all together, we get

$$x(e^x-1) = (e^x-x-1)+a_1x(e^x-1)+a_2x^2(e^x-1)+a_3\frac{x^3}{2!}(e^x-1)+a_4\frac{x^4}{3!}(e^x-1)+\cdots$$

To clean this up, add x to both sides, and divide through by $e^x - 1$. Notice that you can rewrite $\frac{xe^x}{e^x-1}$ as $\frac{x}{1-e^{-x}}$.

Ask Wolfram Alpha (or another computer program) for the power series for $x/(1 - e^{-x})$ to order 20. Check that it agrees with the numbers you found two weeks ago, and see what the next few numbers would have looked like if you'd kept going.

If you have time to kill, think about what would be involved in proving that the formula you've found for $1^p + 2^p + \cdots + n^p$ is correct.