Worksheet 9

Math 206

Monday, November 20, 2023

On worksheet 6, we let $f_p(n) = 1^p + 2^p + \cdots + n^p$ for any positive integers p and n, saw that it was a polynomial in n of degree $p+1$, inspected the coefficients of $f_1(n)$, $f_2(n)$, and so on up to $f_{14}(n)$, and found a lot of patterns. If you ask a computer to factor the polynomials, you get

$$f_1(n) = \frac{n(n+1)}{2}$$

$$f_2(n) = \frac{n(n+1)(2n+1)}{6}$$

$$f_3(n) = \frac{n^2(n+1)^2}{4}$$

$$f_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$f_5(n) = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

$$f_6(n) = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$$

$$f_7(n) = \frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24}$$

$$f_8(n) = \frac{n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)}{90}$$

$$f_9(n) = \frac{n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)}{20}$$

$$f_{10}(n) = \frac{n(n+1)(2n+1)(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)}{66}$$

$$f_{11}(n) = \frac{n^2(n+1)^2(2n^8+8n^7+4n^6-16n^5-5n^4+26n^3-3n^2-20n+10)}{24}$$

and then f_{12}, f_{13}, and f_{14} are too wide for the page. The first two or three of these are more or less familiar by now. What obvious patterns do you see in the later ones? What doesn't seem to follow any pattern?

Of course $1^p + 2^p + \cdots + n^p$ doesn't make sense if n is negative. But the polynomials do, and they all have $n+1$ as a factor, so they have -1 as a root: that is, $f_p(-1) = 0$. What's up with that? More generally, what's up with $f_p(-2)$, $f_p(-3)$, $f_p(-4)$, and so on? Find a few values (maybe for $p = 1$ and $p = 2$) and see if you can explain where they're coming from.

If p is even then $f_p(n)$ also has $-\frac{1}{2}$ as a root. What's up with that?

If you have time to kill, revisit the first page of Worksheet 8, but rather than finding $a_1, a_2, a_3, \ldots$ using the fact that $f_p(1) = 1$ for all p, use the fact that $f_p(-1) = 0$. Is it nicer? Less nice? About the same?

Or if you prefer, you could muse about why $f_9(n)$ and $f_{10}(n)$ have $n^2 + n - 1$ as a factor. This implies that they have $-\varphi$ as a root, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. I don't know if it has some deeper meaning or if it's just a weird coincidence. I checked up to $p = 100$ and weren't any other values of p for which $f_p(n)$ had extra factors beyond the ones you know about.

If you want to change topics for your final project, you could ask a computer to polt the real and complex roots of these polynomials for larger and larger values of p, and discuss any patterns that you observe.