Solutions to Midterm Exam 1

Each part is worth 10 points.

1.

(a) Sketch the region in the \(xy\)-plane that lies above the \(x\)-axis and below the parabola \(y = 4 - x^2\).

\[
\text{Solution:}
\]

(b) Find the area of the region.

\[
\text{Solution:}
\]

\[
\int_{-2}^{2} (4 - x^2) \, dx = \left[ 4x - \frac{1}{3}x^3 \right]_{-2}^{2} = \left( 8 - \frac{8}{3} \right) - \left( -8 + \frac{8}{3} \right) = \frac{32}{3}.
\]

Alternatively you can find \( \int_{0}^{2} (4 - x^2) \, dx \) and then double it. Or you can remember the formula for area under a parabola that appears in Simpson’s rule.
2. Evaluate \( \int \frac{x}{1 - x^2} \, dx \) in three ways:

(a) By substituting \( u = 1 - x^2 \).

Solution: If \( u = 1 - x^2 \) then \( du = -2x \, dx \). Thus

\[
\int \frac{x}{1 - x^2} \, dx = -\frac{1}{2} \int \frac{1}{u} \, du
\]

\[
= -\frac{1}{2} \ln u + C
\]

\[
= -\frac{1}{2} \ln(1 - x^2) + C.
\]

(b) By partial fractions: Factor \( 1 - x^2 \) as \((1 + x)(1 - x)\), then find numbers \( A \) and \( B \) such that

\[
\frac{x}{(1 + x)(1 - x)} = \frac{A}{1 + x} + \frac{B}{1 - x},
\]

then integrate the right-hand side.

Solution: Clearing denominators in the equation (1), we find that

\[
x = A(1 - x) + B(1 + x).
\]

Setting \( x = 1 \) we find that \( B = \frac{1}{2} \). Setting \( x = -1 \) we find that \( A = -\frac{1}{2} \). Thus

\[
\int \frac{x}{1 - x^2} \, dx = -\frac{1}{2} \int \frac{1}{1 + x} \, dx + \frac{1}{2} \int \frac{1}{1 - x} \, dx
\]

\[
= -\frac{1}{2} \ln(1 + x) - \frac{1}{2} \ln(1 - x) + C.
\]

(c) By substituting \( x = \sin t \). Hint: You may need to recall that \( \frac{\sin t}{\cos t} = \tan t \) and \( \int \tan t \, dt = \ln(\sec t) + C = -\ln(\cos t) + C \).

Solution: If \( x = \sin t \) then \( dx = \cos t \, dt \). Thus

\[
\int \frac{x}{1 - x^2} \, dx = \int \frac{\sin t}{\cos^2 t} \cos t \, dt
\]

\[
= \int \frac{\sin t}{\cos t} \, dt
\]

\[
= \int \tan t \, dt
\]

\[
= -\ln(\cos t) + C
\]

\[
= -\ln \sqrt{1 - x^2} + C.
\]
(d) Your answers to parts (a), (b), and (c) look different. Show that they are the same.

Solution: For part (b) we can write

\[-\frac{1}{2} \ln(1 + x) - \frac{1}{2} \ln(1 - x) = -\frac{1}{2} \left( \ln(1 + x) + \ln(1 - x) \right) \]
\[= -\frac{1}{2} \ln((1 + x)(1 - x)) \]
\[= -\frac{1}{2} \ln(1 - x^2) \]

which agrees with part (a). For part (c) we can write

\[-\ln \sqrt{1 - x^2} = -\ln \left( (1 - x^2)^{1/2} \right) \]
\[= -\frac{1}{2} \ln(1 - x^2) \]

which agrees with part (a) again.
3.

(a) Evaluate $\lim_{x \to \infty} \frac{\ln x}{x}$.

Hint: This is of the form $\frac{\infty}{\infty}$, so you can use l'Hôpital's rule.

Solution: By l'Hôpital's rule we have

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

(b) Evaluate $\int x^{-2} \ln x \, dx$.

Hint: Integrate by parts, letting $u = \ln x$ and $dv = x^{-2} \, dx$.

Solution: If $u = \ln x$ and $dv = x^{-2} \, dx$ then $du = x^{-1} \, dx$ and $v = -x^{-1}$.

$$\int x^{-2} \ln x \, dx = \int x^{-1} \ln x + \int x^{-2} \, dx$$

$$= -x^{-1} \ln x - x^{-1} + C.$$ 

(c) Evaluate $\int_{1}^{\infty} x^{-2} \ln x \, dx$.

Hint: Recycle your answers to parts (a) and (b).

Solution:

$$\int_{1}^{\infty} x^{-2} \ln x \, dx = \lim_{B \to \infty} \int_{1}^{B} x^{-2} \ln x \, dx$$

$$= \lim_{B \to \infty} \left[ -x^{-1} \ln x - x^{-1} \right]_{1}^{B}$$

$$= \lim_{B \to \infty} \left( \left( -\frac{\ln B}{B} - \frac{1}{B} \right) - \left( -1 \cdot 0 - 1 \right) \right)$$

$$= (0 - 0) + 1$$

$$= 1.$$ 

(d) Substitute $x = e^t$ into the integral from part (c) and clean it up to get $\int_{0}^{\infty} te^{-t} \, dt$. On the practice exam you found that the latter equals 1. Does this agree with your answer to part (c)?

Solution: If $x = e^t$ then $dx = e^t \, dt$. Solving $x = e^t$ for $t$ we get $t = \ln x$; thus if $x = 1$ then $t = 0$, and as $x \to \infty$ we have $t \to \infty$.

$$\int_{1}^{\infty} x^{-2} \ln x \, dx = \int_{0}^{\infty} e^{-2t} \cdot t \cdot e^t \, dt$$

$$= \int_{0}^{\infty} te^{-t} \, dt.$$ 

For part (c) we got 1, so this agrees with the practice exam.