Solutions to Practice Midterm 1

1.

(a) Sketch the region in the xy-plane whose area is given by \( \int_0^1 \sqrt{4 - x^2} \, dx \).

Solution:

(b) Evaluate the integral geometrically by thinking about triangles and wedges of circles.

Solution: The area of a circle of radius 2 is \( 4\pi \). The wedge shown is a twelfth of a circle. The triangle shown has a base of 1 and a height of \( \sqrt{3} \). Thus the area is \( \frac{\pi}{3} + \frac{\sqrt{3}}{2} \).

(c) Evaluate the integral algebraically by substituting \( x = 2 \sin t \). Does your answer agree with your answer to part (b)? Hint: You will need the half-angle formula \( \cos^2 t = \frac{1 + \cos 2t}{2} \). Further hint: Don’t forget to change bounds: if \( 2 \sin t = 1 \) then \( \sin t = \frac{1}{2} \), so \( t = \ldots \).

Solution: If \( x = 2 \sin t \) then \( dx = 2 \cos t \, dt \). Then we have

\[
4 - x^2 = 4 - 4 \sin^2 t = 4(1 - \sin^2 t) = 4 \cos^2 t,
\]

so \( \sqrt{4 - x^2} = 2 \cos t \). To change bounds, write \( \sin t = x/2 \); if \( x = 0 \) then
\[
\int_0^1 \sqrt{4 - x^2} \, dx = \int_0^{\pi/6} 2 \cos t \cdot 2 \cos t \, dt \\
= \int_0^{\pi/6} 4 \cos^2 t \, dt \\
= \int_0^{\pi/6} (2 + 2 \cos 2t) \, dt \\
= \left[ 2t + \sin 2t \right]_0^{\pi/6} \\
= \pi/3 + \frac{\sqrt{3}}{2} - 0,
\]

where in the third line we have used the given half-angle formula.

2. Evaluate \( \int x \sqrt{x - 1} \, dx \) in two ways:

(a) By substituting \( u = x - 1 \). Hint: Then \( x = u + 1 \).

**Solution:** With this \( u \) we have \( du = dx \).

\[
\int x \sqrt{x - 1} \, dx = \int (u + 1) \sqrt{u} \, du \\
= \int \left( u^{3/2} + u^{1/2} \right) \, du \\
= \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \\
= \frac{2}{5} (x - 1)^{5/2} + \frac{2}{3} (x - 1)^{3/2} + C
\]

(b) By parts, letting \( u = x \) and \( dv = \sqrt{x - 1} \, dx \) then \( du = dx \) and \( v = \frac{2}{3} (x - 1)^{3/2} \).

**Solution:** If \( u = x \) and \( dv = \sqrt{x - 1} \, dx \) then \( du = dx \) and \( v = \frac{2}{3} (x - 1)^{3/2} \).

\[
\int x \sqrt{x - 1} \, dx = \frac{2}{3} x (x - 1)^{3/2} - \int \frac{2}{3} (x - 1)^{3/2} \, dx \\
= \frac{2}{3} x (x - 1)^{3/2} - \frac{2}{3} \cdot \frac{2}{5} (x - 1)^{5/2} + C \\
= \frac{2}{3} x (x - 1)^{3/2} - \frac{4}{15} (x - 1)^{5/2} + C.
\]

(c) Your answers to parts (a) and (b) look different. Show that they are the same. Hint: Factor out \( (x - 1)^{3/2} \) and clean up what’s left. **Solution:**

For part (a) we have

\[
\frac{2}{5} (x - 1)^{5/2} + \frac{2}{3} (x - 1)^{3/2} = (x - 1)^{3/2} \left( \frac{2}{5} (x - 1) + \frac{2}{3} \right) \\
= (x - 1)^{3/2} \left( \frac{2}{3} x + \frac{4}{15} \right).
\]
For part (b) we have
\[
\frac{3}{2} x(x - 1)^{3/2} - \frac{4}{15} (x - 1)^{5/2} = (x - 1)^{3/2} \left( \frac{3}{2} x - \frac{4}{15} (x - 1) \right)
= (x - 1)^{3/2} \left( \frac{6}{5} x + \frac{4}{15} \right)
\]

3.

(a) What is \( \lim_{t \to \infty} e^t \)? What is \( \lim_{t \to \infty} e^{-t} \)? What is \( \lim_{t \to \infty} te^{-t} \)? Hint: Write the latter as \( \lim_{t \to \infty} \frac{t}{e^t} \), which is of the form \( \frac{\infty}{\infty} \), so you can use l'Hôpital's rule.

**Solution:** First we have \( \lim_{t \to \infty} e^t = \infty \), so \( \lim_{t \to \infty} e^{-t} = 0 \). Then by l'Hôpital's rule we have
\[
\lim_{t \to \infty} \frac{t}{e^t} = \lim_{t \to \infty} \frac{1}{e^t} = 0.
\]

(b) Find \( \int te^{-t} \, dt \). Hint: Integrate by parts.

**Solution:** Let \( u = t \) and \( dv = e^{-t} \, dt \), so \( du = dt \) and \( v = -e^{-t} \).
\[
\int te^{-t} \, dt = -te^{-t} + \int e^{-t} \, dt
= -te^{-t} - e^{-t} + C.
\]

(c) Find \( \int_0^\infty te^{-t} \, dt \). Hint: Recycle your answer to parts (a), (b), and especially (c).

**Solution:**
\[
\int_0^\infty te^{-t} \, dt = \lim_{B \to \infty} \int_0^B te^{-t} \, dt
= \lim_{B \to \infty} \left[ -te^{-t} - e^{-t} \right]_0^B
= \lim_{B \to \infty} \left( (-Be^{-B} - e^{-B}) - (-0 \cdot 1 - 1) \right)
= (0 - 0) + 1
= 1.
\]