Solutions to Practice Final

1.

(a) \( \int \frac{1}{2 - 3x} \, dx \) is which of the following?

- \( \ln(2 - 3x) + C \)
- \( -\ln(2 - 3x) + C \)
- \( 3 \ln(2 - 3x) + C \)
- \( -3 \ln(2 - 3x) + C \)
- \( \frac{1}{3} \ln(2 - 3x) + C \)
- \( -\frac{1}{3} \ln(2 - 3x) + C \)
- \( \frac{1}{2} \ln(2 - 3x) + C \)
- \( -\frac{1}{2} \ln(2 - 3x) + C \)

**Solution:** It is \(-\frac{1}{3} \ln(2 - 3x) + C\). By the chain rule, the derivative of \(\ln(2 - 3x)\) is \(\frac{1}{2 - 3x} \cdot (-3)\), so we need the \(-\frac{1}{3}\) out front to get \(-\frac{1}{3} \ln(2 - 3x)\). Alternatively, if we let \(u = 2 - 3x\) then \(du = -3 \, dx\), so \(dx = -\frac{1}{3} \, du\).

(b) \( \int \frac{1}{(2 - 3x)^2} \, dx \) is which of the following? Why?

- \( \ln \left( (2 - 3x)^2 \right) + C \)
- \( -\ln \left( (2 - 3x)^2 \right) + C \)
- \( \frac{1}{2 - 3x} + C \)
- \( -\frac{1}{2 - 3x} + C \)
- \( 3 \ln \left( (2 - 3x)^2 \right) + C \)
- \( -3 \ln \left( (2 - 3x)^2 \right) + C \)
- \( 3 \cdot \frac{1}{2 - 3x} + C \)
- \( -3 \cdot \frac{1}{2 - 3x} + C \)
- \( \frac{1}{3} \ln \left( (2 - 3x)^2 \right) + C \)
- \( -\frac{1}{3} \ln \left( (2 - 3x)^2 \right) + C \)
- \( \frac{1}{3} \cdot \frac{1}{2 - 3x} + C \)
- \( -\frac{1}{3} \cdot \frac{1}{2 - 3x} + C \)
- \( 2 \ln \left( (2 - 3x)^2 \right) + C \)
- \( -2 \ln \left( (2 - 3x)^2 \right) + C \)
- \( 2 \cdot \frac{1}{2 - 3x} + C \)
- \( -2 \cdot \frac{1}{2 - 3x} + C \)
- \( \frac{1}{2} \ln \left( (2 - 3x)^2 \right) + C \)
- \( -\frac{1}{2} \ln \left( (2 - 3x)^2 \right) + C \)
- \( \frac{1}{2} \cdot \frac{1}{2 - 3x} + C \)
- \( -\frac{1}{2} \cdot \frac{1}{2 - 3x} + C \)

**Solution:** It is \(\frac{1}{3} \ln \left( (2 - 3x)^2 \right) + C\). By the chain rule, the derivative of \((2 - 3x)^{-1}\) is \(-(2 - 3x)^{-2} \cdot (-3)\), and the minus signs cancel, so we need the \(\frac{1}{3}\) out front to get \(\frac{1}{3} \ln \left( (2 - 3x)^2 \right)\). Alternatively, if we let \(u = 2 - 3x\) then \(du = -3 \, dx\), so \(dx = -\frac{1}{3} \, du\); then we also need to note that \(\int u^{-2} \, du = -u^{-1} + C\).
2. (Based on §6.3 #16.)

(a) Sketch the region in the plane bounded by the parabolas \( y = x^2 \) and \( y = 2 - x^2 \). Find the coordinates of the points where the two curves meet, and the points where they meet the \( y \)-axis.

**Solution:**

(b) Sketch the solid obtained by revolving the region around the \( y \)-axis.

**Solution:**

(c) Find the volume of the solid using the shell method.

**Solution:** A typical shell looks like this:

Its radius is \( x \) and its height is \((2 - x^2) - (x^2) = 2 - 2x^2\), so its area is \(2\pi x \cdot (2 - 2x^2) = 4\pi (x - x^3)\), so the volume of the solid is

\[
4\pi \int_0^1 (x - x^3) \, dx = 4\pi \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 4\pi \left( \frac{1}{2} - \frac{1}{4} - 0 \right) = \pi.
\]

Note that the integral needs to go from 0 to 1, not from \(-1\) to 1, so that we only get each shell once.

(d) Find the volume of the solid using the disc method. Hint: It may be convenient to find the volume of just the top half or the bottom half, and then double it.

**Solution:** We take the hint. A typical shell in the bottom half looks like this:

It radius is \( \sqrt{y} \), so its area is \( \pi y \), so the volume of the bottom half of the solid is

\[
\pi \int_0^1 y \, dy = \pi \left[ \frac{y^2}{2} \right]_0^1 = \pi \left( \frac{1}{2} - 0 \right) = \frac{\pi}{2}.
\]

Doubling this we get \( \pi \) again.
3. (§7.4 #13. Seasonal.) A turkey is taken out of the oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. Newton’s law of cooling states that the turkey cools at a rate proportional to the difference between its temperature and the temperature of the room.

(a) If the temperature of the turkey is 150°F after half an hour, what will it be after 45 minutes?

**Solution:** Let \( t \) be the time in minutes and \( y \) the temperature of the turkey. By Newton’s law of cooling,

\[
\frac{dy}{dt} = k(y - 75)
\]

for some number \( k \). Separating variables, we get

\[
\frac{1}{y - 75} \, dy = k \, dt.
\]

Integrating, we get

\[
\ln(y - 75) = kt + C
\]

for some number \( C \). Exponentiating, we get

\[
y - 75 = e^{kt}e^C.
\]

We rename \( e^C \) to \( A \) and move the 75 to the other side to get

\[
y = 75 + Ae^{kt}.
\]

Next, we know that when \( t = 0 \), \( y = 185 \), so we find that \( A = 110 \), so

\[
y = 75 + 110e^{kt}.
\]

Next, we know that when \( t = 30 \), \( y = 150 \), so we find that \( k = \frac{1}{30} \ln\left(\frac{75}{110}\right) \), so

\[
y = 75 + 110e^{t/30 \ln(75/110)}.
\]

Finally, we plug in \( t = 45 \) and find that \( y \approx 137 \)° F.

(b) When will the turkey have cooled to 100°F?

**Solution:** Letting \( y = 100 \) in the last equation, we find that

\[
t = 30 \cdot \frac{\ln(25/110)}{\ln(75/110)} \approx 116 \text{ minutes}.
\]

So you have plenty of time to make the gravy before the bird goes cold.
4. (Based on an example we did in lecture.) According to Toricelli's law, water drains from a tank at a rate proportional to the square root of the depth. A spherical tank of radius 1 meter was initially full of water and took an hour to drain. At what time was the tank half full?

Let's all choose the same variables: let \( t \) be the time in minutes, \( y \) the depth of the water in meters, and \( V \) the volume of water in cubic meters.

(a) Draw a picture of the tank partly filled with water, but not exactly half full. Label \( y \).

Solution:

(b) What is the area of the surface of the water when the depth is \( y \)? Hint: The surface is a circle; start by finding its radius. It may help to draw a cross-section of the tank and some kind of triangle.

Solution: Here is the cross-section:

When the depth is \( y \), the height of the right triangle is \( (1 - y) \), and the hypotenuse is 1, so the base is

\[
\sqrt{1 - (1 - y)^2} = \sqrt{1 - (1 - 2y + y^2)} = \sqrt{2y - y^2}.
\]

This is the radius of the surface of the water. Thus the area is \( \pi(\sqrt{2y - y^2})^2 = \pi(2y - y^2) \).

(c) Write a differential equation expressing Toricelli's law. Hint: The left-hand side is \( dV/dt \); the right-hand side involves \( y \) and a constant \( k \).

Solution: The rate at which the volume of water is changing is proportional to the square root of the depth, so

\[
\frac{dV}{dt} = k\sqrt{y}
\]

for some constant \( k \).

(d) You could find \( V \) as a function of \( y \), or \( y \) as a function of \( V \), but it would be a lot of work. Instead, observe that \( dV/dy \) is the area you found in part (b) – why is this true? Then observe that

\[
\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt}
\]

by the chain rule, so you can replace \( dV/dt \) in your equation from part (c) with your new expression for \( dV/dy \), times \( dy/dt \), to get a differential equation involving only \( y \).

Solution: \( dV/dy \) is the area of the surface of the water, because if the depth \( y \) increases a tiny bit \( dy \) then \( V \) increases by the area times \( dy \). Alternatively you can say it's the fundamental theorem of calculus: the volume is the integral of the areas of the slices, so the derivative of the volume is the area. Using the chain rule observation, our differential equation becomes

\[
\pi(2y - y^2) \frac{dy}{dt} = k\sqrt{y}.
\]
(e) Take your equation from part (d), separate variables, and integrate. But do not try to solve for $y$.

**Solution:** First we separate variables:

$$\frac{\pi}{2} y - \frac{y^2}{\sqrt{y}} \, dy = k \, dt.$$

We clean this up:

$$\pi(2y^{1/2} - y^{3/2}) \, dy = k \, dt.$$

Then we integrate:

$$\pi \left( \frac{4}{3} y^{3/2} - \frac{2}{3} y^{5/2} \right) = kt + C.$$

(f) What is $y$ when $t = 0$? When $t = 60$? Use these facts to find $k$ and $C$. They will be a little messy, but not horribly so.

**Solution:** When $t = 0$, $y = 2$. We plug this into our equation, noting that $2^{3/2} = 2\sqrt{2}$ and $2^{5/2} = 4\sqrt{2}$, to get

$$C = \frac{16\pi \sqrt{2}}{15}.$$

When $t = 60$, $y = 0$. We plug this into our equation to get

$$k = -\frac{1}{60} \frac{16\pi \sqrt{2}}{15}.$$

(g) What is $y$ when the tank is half full? Find $t$ corresponding to that value $y$. It should be a little less than 30 minutes, because the water flows faster at first and slower later on.

**Solution:** When the tank is half full, $y = 1$. Plugging this into our equation, we get

$$\frac{14\pi}{15} = -\frac{1}{60} \frac{16\pi \sqrt{2}}{15} t + \frac{16\pi \sqrt{2}}{15}.$$

Dividing through by $\frac{16\pi \sqrt{2}}{15}$, this becomes

$$\frac{14}{16\sqrt{2}} = -\frac{1}{60} t + 1.$$

We can clean up the left-hand side slightly to get $\frac{7\sqrt{2}}{16}$. We solve for $t$ to get

$$t = 60 \cdot \left( 1 - \frac{7\sqrt{2}}{16} \right) \approx 23 \text{ minutes},$$

which is indeed a little less than 30.