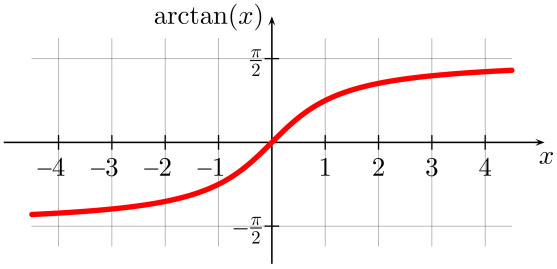
Solution to Quiz 7

Before the quiz, we established that

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C,$$

and we graphed $\arctan x$:



In particular we noted that

$$\arctan(0) = 0 \qquad \arctan(\pm 1) = \pm\frac{\pi}{4} \qquad \lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}.$$

Then the quiz was §6.8 #5: Let $f(x) = \dfrac{c}{1+x^2}$.

(a) For what value of c is f a probability density function?

(b) For that value of c, what is the probability that x is between -1 and 1?

For (a), we want to find the value of c for which $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$. We calculate:

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = c\arctan x\Big|_{-\infty}^{\infty} = c\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = c\cdot\pi.$$

If $c\cdot\pi = 1$ then $c = 1/\pi$. For (b), the answer is

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{1/\pi}{1+x^2}\,dx = \frac{1}{\pi}\arctan x\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{1}{2}.$$