Solutions to Final Exam

1. §6.6 #11: A heavy rope, 30 ft long, weights 0.6 lb/ft and hangs over the edge of a building 70 ft high. How much work is done in pulling the rope to the top of the building?

   (Recall that work = force × distance.)

   (a) Write an integral to calculate the work.
   Solution: Consider the bit of rope that is x feet from the top of the building and dx feet long. This piece of rope weighs $0.6\,dx$ pounds, and we are lifting it x feet, doing $0.6x\,dx$ foot-pounds of work. Thus the total work is

   $$\int_0^{30} 0.6x\,dx.$$

   (b) Evaluate the integral.
   Solution:

   $$\begin{aligned}\int_0^{30} 0.6x\,dx &= 0.3\,x^2\Big|_0^{30}\\ &= 0.3\cdot 30^2\\ &= 270 \text{ ft}\cdot\text{lbs}.\end{aligned}$$

   Now suppose that after hauling up half the rope, you stop, leaving 15 feet of rope hanging over the edge of the building. How much work have you done?

   (c) Write an integral to calculate the work.
   Solution: For a bit of rope that is $x \le 15$ feet from the top, we are lifting it x feet, doing $0.6x\,dx$ foot-pounds of work. For a bit that is $x \ge 15$ feet from the top, we are only lifting it 15 feet, doing $.6\cdot 15\,dx = 9\,dx$ foot-pounds of work. Thus the total work is

   $$\int_0^{15} 0.6x\,dx + \int_{15}^{30} 9\,dx.$$

(d) Evaluate the integral.

Solution:

$$\begin{aligned}\int_0^{15} 0.6x\,dx + \int_{15}^{30} 9\,dx &= 0.3\,x^2\Big|_0^{15} + 9x\Big|_{15}^{30} \\ &= 0.3\cdot 15^2 + 9\cdot(30-15) \\ &= 202.5\ \text{ft}\cdot\text{lbs},\end{aligned}$$

which is substantially more than half of 270 ft·lbs!

2. §6.6 #6: A spring has a natural length of 15 cm. If a force of 25 N is required to keep it stretched to a length of 25 cm, how much work is required to stretch it from 15 cm to 20 cm?

   (According to Hooke's law, the force required to hold a stretched spring is proportional to how far it has been stretched beyond its natural length; that is, proportional to the stretched length minus the natural length.)

   (a) Write an integral to calculate the work.

   Solution: Let x be the length of the spring in centimeters. Hooke's law says that $F = k(x-15)$ for some constant k. We know that $F = 25$ when $x = 25$, so $25 = 10k$, so $k = 2.5$.

   Now start from 15 cm and stretch to 20cm. When the length is x, we exert $2.5(k-15)$ Newtons of force for a distance of dx centimeters, doing $2.5(x-15)\,dx$ N·cm of work. Thus the total work is

   $$\int_{15}^{20} 2.5(x-15)\,dx.$$

   (b) Evaluate the integral.

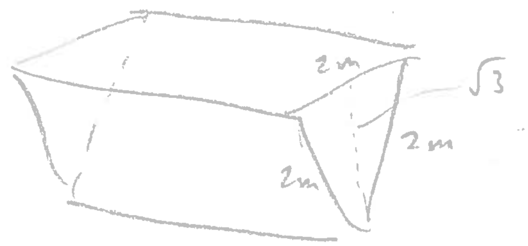
   Solution: It is convenient to substitute $u = x - 15$, so $du = dx$. Physically, u is how far the spring has been stretched beyond its natural length.

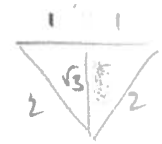
   $$\begin{aligned}\int_{x=15}^{x=20} 2.5(x-15)\,dx &= \int_{u=0}^{u=5} 2.5u\,du \\ &= 1.25\,u^2\Big|_0^5 \\ &= 31.25\ \text{N}\cdot\text{cm}.\end{aligned}$$

3. §6.6 #37: A trough is filled with a liquid of density 1030 kg/m^3. The ends of the trough are equilateral triangles with sides 2 m long and with the vertex at the bottom.

(a) Draw a rough picture of the trough. How deep is it?

Solution:

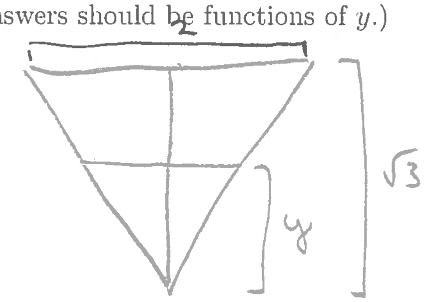




The trough is $\sqrt{3}$ m deep.

(b) Draw a triangular cross-section of the trough, and a horizontal line at a height y above the bottom of the trough. How wide is the line? What is the distance from the top of the trough down to the line? (Both answers should be functions of y.)

Solution:



By similar triangles, $\frac{\text{width}}{y} = \frac{2}{\sqrt{3}}$, so the width is $\frac{2}{\sqrt{3}}y$.

The distance from the top down to the line is $\sqrt{3} - y$.

(c) Write an integral to calculate the hydrostatic force on one end of the trough. *Do not evaluate the integral.* (Recall that pressure = density × 9.8 × depth, that force = pressure × area, and that the area of this horizontal slice is its width times dy.)

Solution:

$$\int_0^{\sqrt{3}} 1030 \cdot 9.8 \cdot (\sqrt{3} - y) \cdot \frac{2}{\sqrt{3}} y \, dy.$$

4. §7.3 #15: When a cold drink is taken from a refrigerator, its temperature is 40°F. After 25 minutes in a 70°F room, its temperature has increased to 50°F.

   (According to Newton's law of cooling, the rate at which the drink warms up is proportional to the difference between the temperature of the room and the temperature of the drink.)

   (a) What is the temperature of the drink after 50 minutes?
   Solution: Let y be the temperature of the drink in °F, and t the time in minutes. By Newton's law of cooling we have

   $$\frac{dy}{dt} = k(70 - y),$$

   for some constant k. Separate variables to get

   $$\frac{dy}{70 - y} = k\,dt.$$

   Integrate to get

   $$-\ln(70 - y) = kt + C$$

   for some constant C. Plug in $t = 0$, $y = 40$ to get $C = -\ln(30)$. Thus

   $$-\ln(70 - y) = kt - \ln(30).$$

   Plug in $t = 25$, $y = 50$ to get

   $$-\ln(20) = 25k - \ln(30)$$

   and thus

   $$k = \frac{\ln(30) - \ln(20)}{25} \approx .0162.$$

   Thus

   $$-\ln(70 - y) = .0162\,t - \ln(30).$$

   Now plug in $t = 50$:

   $$-\ln(70 - y) = .0162 \cdot 50 - \ln(30) \approx -2.59.$$

   Multiply through by -1 and exponentiate to get

   $$70 - y = e^{2.59}.$$

   Solve for y to get

   $$y = 70 - e^{2.59} \approx 56.7°\mathrm{F}.$$

(b) When will the temperature of the drink be 60°F?

Solution: Again we have

$$-\ln(70-y) = .0162\,t - \ln(30).$$

Plug in $y = 60$ to get

$$-\ln(10) = 0.162\,t - \ln(30).$$

Solve for t to get

$$t = \frac{\ln(30) - \ln(10)}{0.162} \approx 67.8 \text{ minutes}.$$

5. §7.2 #45: The volume of a room is 6000 ft^3. The air in the room contains 0.15% carbon dioxide initially. Fresher air containing only 0.05% carbon dioxide flows in through a window at a rate of 60 ft^3/min. Well-mixed air flows out another window at the same rate, 60 ft^3/min.

(a) Let y be the amount of carbon dioxide in the room in ft^3, and let t be the time in minutes. Explain why $y = 9$ when $t = 0$.

Solution: Becuase 0.15% of 6000 is 9.

(b) How many ft^3/min of carbon dioxide are entering the room? (This is a number.)

Solution: 0.05% of 60 ft^3/min, which is

$$0.03 \text{ ft}^3/\text{min}.$$

(c) How many ft^3/min of carbon dioxide are leaving the room? (This depends on y.)

Solution: Every minute, 60 ft^3 of air is going out the window. This is 1% of the air in the room; thus the amount of carbon dioxide leaving the room is 1% of y:

$$0.01\,y \text{ ft}^3/\text{min}.$$

(d) Write a differential equation ($\frac{dy}{dt} = \cdots$) that describes how the amount of carbon dioxide in the room is changing.

Solution:

$$\frac{dy}{dt} = 0.03 - 0.01y.$$

(e) What is the percentage of carbon dioxide in the room after an hour?

Solution: Separate variables to get

$$\frac{dy}{0.03 - 0.01y} = dt.$$

Integrate to get

$$-100 \ln |0.03 - 0.01y| = t + C.$$

Plug in $t = 0$, $y = 9$ to get

$$C = -100 \ln |-0.06| \approx 281.34.$$

Notice that the absolute value flips the sign of the -0.06; the same thing will have to happen again when we go to solve for y at $t = 60$.

So we have

$$-100 \ln |0.03 - 0.01y| = t + 281.34.$$

Plug in $t = 60$ and solve for y:

$$\begin{aligned} -100 \ln |0.03 - 0.01y| &= 341.34 \\ \ln |0.03 - 0.01y| &= -3.4134 \\ |0.03 - 0.01y| &= e^{-3.4134} \approx 0.03293. \end{aligned}$$

From our discussion of signs a moment ago, we know that the absolute value needs to flip the sign of $0.03 - 0.01y$; indeed, $0.03 - 0.01y$ is the rate at which the volume of CO_2 is changing, and for physical reasons we know this is negative. Thus

$$\begin{aligned} -0.03 + 0.01y &= 0.03293 \\ 0.01y &= 0.06293 \\ y &= 6.293 \text{ ft}^3. \end{aligned}$$

This is 0.1049% of the air in the room, so we're slowly moving toward 0.05%, but it would help to open some more windows.