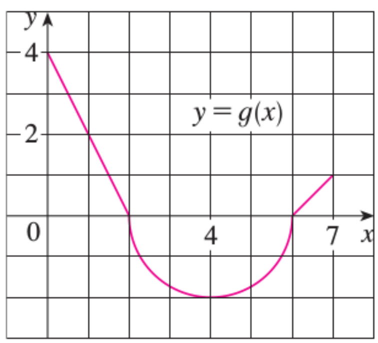
Solutions to Midterm 1

1. §5.2 #32: The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.



   (a) $\displaystyle\int_0^2 g(x)\,dx$
   Solution: It's a triangle with base = 2 and height = 4, so the area is 4.

   (b) $\displaystyle\int_2^6 g(x)\,dx$
   Solution: It's a semi-circle with radius = 2; the area of the whole circle would be $\pi \cdot 2^2 = 4\pi$, so the area of the semi-circle is 2π. But it's below the x-axis, so the integral is -2π.

   (c) $\displaystyle\int_0^7 g(x)\,dx$
   Solution: It's the sum previous two, plus a triangle with area $1/2$, so the integral is $4 - 2\pi + 1/2$.

2. §5.3 #7: Evaluate
$$\int_{-1}^{0} (2x - e^x)\,dx.$$

Solution:
$$\begin{aligned}
\int_{-1}^{0} (2x - e^x)\,dx &= \Big[x^2 - e^x\Big]_{-1}^{0} \\
&= (0-1) - (1 - e^{-1}) \\
&= \frac{1}{e} - 2.
\end{aligned}$$

3. §5.5 #44: Evaluate
$$\int_{0}^{\sqrt{\pi}} x\cos(x^2)\,dx.$$

Solution: Let $u = x^2$, so $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$. Then
$$\begin{aligned}
\int_{x=0}^{x=\sqrt{\pi}} x\cos(x^2)\,dx &= \int_{u=0}^{u=\pi} \cos u \cdot \tfrac{1}{2}du \\
&= \tfrac{1}{2}\sin u\,\Big|_{0}^{\pi} \\
&= \tfrac{1}{2}\cdot 0 - \tfrac{1}{2}\cdot 0 \\
&= 0.
\end{aligned}$$

4. §5.5 #55: Evaluate
$$\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}.$$

Hint: Substitute $u = \ln x$.

Solution: Let $u = \ln x$, so $du = 1/x\,dx$. Then
$$\begin{aligned}
\int_{x=e}^{x=e^4} \frac{dx}{x\sqrt{\ln x}} &= \int_{u=1}^{u=4} \frac{du}{\sqrt{u}} \\
&= \int_{1}^{4} u^{-1/2}\,du \\
&= 2u^{1/2}\Big|_{1}^{4} \\
&= 2\cdot 2 - 2\cdot 1 \\
&= 2.
\end{aligned}$$

5. §5.6 #18: Evaluate
$$\int_4^9 \frac{\ln y}{\sqrt{y}}\, dy.$$
Hint: Integrate by parts, letting $u = \ln y$ and $dv = y^{-1/2}\, dy$.

Solution: With u and v as in the hint, we have $du = y^{-1}\, dy$ and $v = 2y^{1/2}$. Thus
$$\begin{aligned}
\int_4^9 \frac{\ln y}{\sqrt{y}}\, dy &= \Big[\ln y \cdot 2y^{1/2}\Big]_4^9 - \int_4^9 2y^{-1/2}\, dy \\
&= 6\ln 9 - 4\ln 4 - 4y^{1/2}\Big|_4^9 \\
&= 6\ln 9 - 4\ln 4 - (4\cdot 3 - 4\cdot 2) \\
&= 12\ln 3 - 8\ln 2 - 4.
\end{aligned}$$
(In the first line, note that $v\, du = 2y^{1/2}\cdot y^{-1}\, dy = 2y^{-1/2}\, dy$.)

6. §5.7 #3: Evaluate
$$\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\, dx.$$
Hint: Use the identity $\cos^2 x = 1 - \sin^2 x$ and substitute $u = \sin x$. Alternatively you could use $\sin^2 x = 1 - \cos^2 x$ and substitute $u = \cos x$, but it will be messier.

Solution: Following the hint, we write
$$\begin{aligned}
\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\, dx &= \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x\, dx \\
&= \int_{\pi/2}^{3\pi/4} \sin^5 x (1 - \sin^2 x) \cos x\, dx.
\end{aligned}$$

Then we let $u = \sin x$, so $du = \cos x\, dx$, so this becomes

$$\begin{aligned}
\int_1^{\sqrt{2}/2} u^5(1-u^2)\, du &= \int_1^{\sqrt{2}/2} (u^5 - u^7)\, du \\
&= \left[\frac{u^6}{6} - \frac{u^8}{8}\right]_1^{\sqrt{2}/2} \\
&= \left(\frac{2^3/2^6}{6} - \frac{2^4/2^8}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) \\
&= -\frac{11}{384}.
\end{aligned}$$

7. §5.7 #21: Use partial fractions to evaluate

$$\int \frac{5x+1}{(2x+1)(x-1)}\, dx.$$

Solution: We wish to find numbers A and B such that

$$\begin{aligned}
\frac{5x+1}{(2x+1)(x-1)} &= \frac{A}{2x+1} + \frac{B}{x-1} \\
&= \frac{Ax - A + 2Bx + B}{(2x+1)(x-1)}.
\end{aligned}$$

Thus we should solve

$$\begin{aligned}
A + 2B &= 5 \\
-A + B &= 1.
\end{aligned}$$

Adding the two lines we get $3B = 6$, so $B = 2$, and thus $A = 1$. Thus

$$\begin{aligned}
\int \frac{5x+1}{(2x+1)(x-1)}\, dx &= \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right) dx \\
&= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.
\end{aligned}$$

8. §5.8 #9: Evaluate
$$\int_4^\infty e^{-y/2}\,dy.$$

Solution: On the one hand, we could say

$$\begin{aligned}\int_4^\infty e^{-y/2}\,dy &= -2e^{-y/2}\Big|_4^\infty\\ &= 0 - -2e^{-2}\\ &= \frac{2}{e^2}.\end{aligned}$$

If you're not comfortable finding an anti-derivative for $e^{-y/2}$ in one step, you could substitute $u = -y/2$, so $dy = -dy/2$, so $dy = -2\,du$:

$$\begin{aligned}\int_4^\infty e^{-y/2}\,dy &= \int_{-2}^{-\infty} e^u(-2\,du)\\ &= \int_{-\infty}^{-2} 2e^u du\\ &= 2e^{-2} - 0\\ &= \frac{2}{e^2}.\end{aligned}$$