Solutions to Midterm 2

You may use a page of notes. Each part is worth 3 points.

1. (a) Based on §6.1 #8. Sketch the region bounded the line $y = x + 4$ and the parabola $y = x^2 - 2x$.

   Solution:

   \begin{center}
   \includegraphics[width=0.5\textwidth]{solutions.png}
   \end{center}

   (b) Find the two points where the line and the parabola intersect.

   Solution:

   \[
   x^2 - 2x = x + 4 \\
   x^2 - 3x - 4 = 0 \\
   (x - 4)(x + 1) = 0 \\
   x = -1 \text{ or } 4.
   \]

   Thus the points are $(-1, 3)$ and $(4, 8)$.

   (c) Set up an integral to find the area of the region. \textit{Do not evaluate the integral}.

   Solution:

   \[
   \int_{-1}^{4} ((x + 4) - (x^2 - 2x)) \, dx = \int_{-1}^{4} (4 + 3x - x^2) \, dx.
   \]
(d) The perimeter of the region consists of a line segment and a segment of a parabola. The length of the line segment is $5\sqrt{2}$: this doesn’t require any calculus. Set up the integral to find the length of the parabolic segment. Do not evaluate the integral.

**Solution:** The parabola is $y = x^2 - 2x$, so $y' = 2x - 2$, so the arc length is

$$
\int_{x=-1}^{x=4} \sqrt{1 + (2x - 2)^2} \, dx = \int_{-1}^{1} \sqrt{4x^2 - 8x + 5} \, dx.
$$

(e) If you had evaluated the integral from part (c), you would have gotten $125/6$. Set up an integral to find the $x$-coordinate of the center of mass of the region. Do not evaluate the integral.

**Solution:**

$$
\int_{-1}^{1} x(4 + 3x - x^2) \, dx = \frac{125}{6}.
$$

(f) Extra credit: Set up an integral to find the $y$-coordinate of the center of mass of the region. Do not evaluate the integral. Hint: If $y = x^2 - 2x$ then $x = 1 \pm \sqrt{y + 1}$.

**Solution:** We need to treat portion of the region below the line $y = 3$ differently from the portion above.

$$
\int_{-1}^{3} y \left(1 + \sqrt{y + 1} \right) \, dy + \int_{3}^{8} y \left(1 + \sqrt{y + 1} \right) \, dy
$$

$$
- \int_{-1}^{3} y \left(1 - \sqrt{y + 1} \right) \, dy + \int_{3}^{8} y \left(1 + \sqrt{y + 1} \right) \, dy
$$

$$
= \frac{125}{6}.
$$
2. On the board you see a sketch of the region that lies above the line $y = 1$ and inside the circle $x^2 + y^2 = 2$.

(a) Roughly sketch the solid obtained by revolving the region around the $x$-axis.

Solution:

(b) Set up the integral to find the volume of the solid from part (a) using disks/washers. Hint: The top half of the circle can be described as $y = \sqrt{2 - x^2}$.

Solution:

$$
\int_{-1}^{1} \left( \pi \left( \sqrt{2 - x^2} \right)^2 - \pi \cdot 1^2 \right) \, dx = \int_{-1}^{1} \pi \left( 1 - x^2 \right) \, dx,
$$

or

$$
2 \int_{0}^{1} \text{(same thing)} \, dx.
$$

(c) Evaluate the integral from part (b).

$$
2\pi \int_{0}^{1} \left( 1 - x^2 \right) \, dx = 2\pi \left[ x - \frac{1}{3} x^3 \right]_{0}^{1} = \frac{4}{3} \pi.
$$

(d) Set up the integral to find the volume of the solid from part (a) using cylindrical shells. Hint: The right half of the circle can be described as $x = \sqrt{2 - y^2}$.

Solution:

$$
\int_{1}^{\sqrt{2}} 2\pi y \cdot 2 \sqrt{2 - y^2} \, dy.
$$

The factor of 2 in front of the square root is because the horizontal slices extend from the left side of the circle to the right side, not just from the $y$-axis to the right side.
(e) Evaluate the integral in part (d). This should agree with your answer to part (c). Hint: Substitute \( u = 2 - y^2 \).

**Solution:** Following the hint, we let \( u = 2 - y^2 \), so \( du = -2y\,dy \).

\[
\int_{y=1}^{y=\sqrt{2}} 4\pi y\sqrt{2 - y^2} \, dy = \int_{u=1}^{u=0} -2\pi \sqrt{u} \, du
\]

\[
= \int_{0}^{1} 2\pi u^{1/2} \, du
\]

\[
= 2\pi \left[ \frac{2}{3} u^{3/2} \right]_{0}^{1}
\]

\[
= \frac{4}{3}\pi.
\]

(f) Roughly sketch the solid obtained by revolving the region around the \( y \)-axis.

**Solution:** It's the top of a sphere.

![Sketch of a sphere](image)

(g) Set up the integral to find the volume of the solid from part (f) solid using cylindrical shells. Hint: The top half of the circle can be described as \( y = \sqrt{2 - x^2} \).

**Solution:**

\[
\int_{0}^{1} 2\pi x \left( \sqrt{2 - x^2} - 1 \right) \, dx
\]

(h) Evaluate the integral from part (g). Hints: Substitute \( u = 2 - x^2 \). Recall that \( 2^{3/2} = 2^1 \cdot 2^{1/2} = 2\sqrt{2} \).
Solution: Following the hint, we let $u = 2 - x^2$, so $du = -2x \, dx$.

\[
\int_{x=0}^{x=1} 2\pi x \left( \sqrt{2 - x^2} - 1 \right) \, dx = \int_{u=2}^{u=1} -\pi \left( \sqrt{u} - 1 \right) \, du
\]

\[
= \int_{1}^{2} \pi \left( u^{1/2} - 1 \right) \, du
\]

\[
= \pi \left[ \frac{2}{3} u^{3/2} - u \right]_{1}^{2}
\]

\[
= \pi \left( \frac{2}{3} \cdot 2\sqrt{2} - 2 - \frac{2}{3} + 1 \right)
\]

\[
= \frac{4\sqrt{2}}{3} \pi - \frac{5}{3} \pi.
\]

(i) Set up the integral to find the volume of the solid from part (f) using disks. Hint: The right half of the circle can be described as $x = \sqrt{2} - y^2$.

Solution:

\[
\int_{1}^{\sqrt{2}} \pi \left( \sqrt{2 - y^2} \right)^2 \, dy
\]

(j) Evaluate the integral in part (i). This should agree with your answer to part (h). Hint: Recall that $(\sqrt{2})^3 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 2\sqrt{2}$.

Solution:

\[
\int_{1}^{\sqrt{2}} \pi \left( 2 - y^2 \right) \, dy = \pi \left[ 2y - \frac{1}{3} y^3 \right]_{1}^{\sqrt{2}}
\]

\[
= \pi \left( 2\sqrt{2} - 1 \cdot 2\sqrt{2} - 2 + \frac{1}{3} \right)
\]

\[
= \frac{4\sqrt{2}}{3} \pi - \frac{5}{3} \pi.
\]

(k) Extra credit: Explain (geometrically) why your answer to part (c) or (e), plus twice your answer to part (h) or (j), plus $2\pi$, should equal $\frac{1}{4} \pi (\sqrt{2})^3 = \frac{8\sqrt{2}}{3} \pi$. Check that your answers satisfy this.

Solution: We can decompose the sphere of radius $\sqrt{2}$ into two “caps,” a “napkin ring,” and a cylinder of radius 1 and height 2, hence of volume $2\pi$. And indeed we have

\[
\frac{4}{3} \pi + \frac{8\sqrt{2}}{3} \pi - \frac{10}{3} \pi + 2\pi = \frac{8\sqrt{2}}{3} \pi.
\]