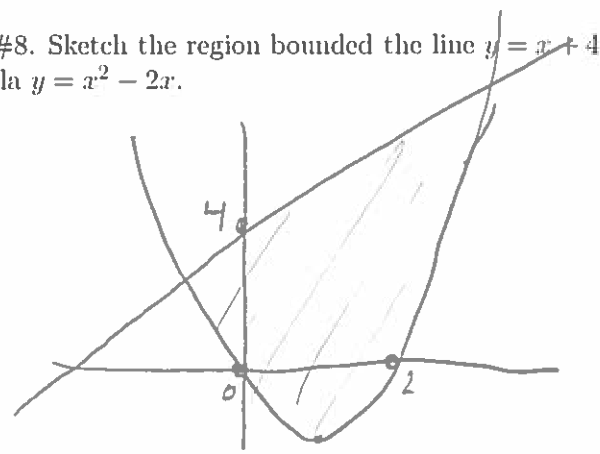
Solutions to Midterm 2

You may use a page of notes. Each part is worth 3 points.

1. (a) Based on §6.1 #8. Sketch the region bounded the line $y = x + 4$ and the parabola $y = x^2 - 2x$.

   Solution:



   (b) Find the two points where the line and the parabola intersect.

   Solution:

$$\begin{aligned} x^2 - 2x &= x + 4 \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x &= -1 \text{ or } 4. \end{aligned}$$

   Thus the points are $(-1, 3)$ and $(4, 8)$.

   (c) Set up an integral to find the area of the region. *Do not evaluate the integral.*

   Solution:

$$\int_{-1}^{4} \left((x+4) - (x^2 - 2x)\right) \, dx = \int_{-1}^{4} \left(4 + 3x - x^2\right) \, dx.$$

(d) The perimeter of the region consists of a line segment and a segment of a parabola. The length of the line segment is $5\sqrt{2}$: this doesn't require any calculus. Set up the integral to find the length of the parabolic segment. *Do not evaluate the integral.*

Solution: The parabola is $y = x^2 - 2x$, so $y' = 2x - 2$, so the arc length is

$$\int_{x=-1}^{x=4} \sqrt{1 + (2x-2)^2}\, dx = \int_{-1}^{4} \sqrt{4x^2 - 8x + 5}\, dx.$$

(e) If you had evaluated the integral from part (c), you would have gotten $125/6$. Set up an integral to find the x-coordinate of the center of mass of the region. *Do not evaluate the integral.*

Solution:

$$\frac{\int_{-1}^{4} x(4 + 3x - x^2)\, dx}{125/6}.$$

(f) Extra credit: Set up an integral to find the y-coordinate of the center of mass of the region. *Do not evaluate the integral.* Hint: If $y = x^2 - 2x$ then $x = 1 \pm \sqrt{y+1}$.

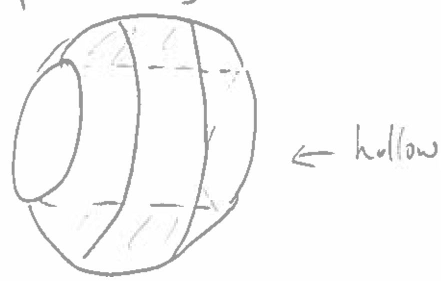
Solution: We need to treat portion of the region below the line $y = 3$ differently from the portion above.

$$\frac{\int_{-1}^{3} y\left((1 + \sqrt{y+1}) - (1 - \sqrt{y+1})\right) dy + \int_{3}^{8} y\left((1 + \sqrt{y+1}) - (y-4)\right) dy}{125/6}.$$

2. On the board you see a sketch of the region that lies above the line $y = 1$ and inside the circle $x^2 + y^2 = 2$.

   (a) Roughly sketch the solid obtained by revolving the region around the x-axis.

   Solution: It's a napkin ring.



   (b) Set up the integral to find the volume of the solid from part (a) using disks/washers. Hint: The top half of the circle can be described as $y = \sqrt{2 - x^2}$.

   Solution:

$$\int_{-1}^{1} \left(\pi \left(\sqrt{2 - x^2} \right)^2 - \pi \cdot 1^2 \right) dx = \int_{-1}^{1} \pi \left(1 - x^2\right) dx,$$

   or

$$2 \int_0^1 \text{(same thing)}\, dx.$$

   (c) Evaluate the integral from part (b).

$$2\pi \int_0^1 \left(1 - x^2\right) dx = 2\pi \left[x - \frac{1}{3} x^3 \right]_0^1 = \frac{4}{3}\pi.$$

   (d) Set up the integral to find the volume of the solid from part (a) using cylindrical shells. Hint: The right half of the circle can be described as $x = \sqrt{2 - y^2}$.

   Solution:

$$\int_1^{\sqrt{2}} 2\pi y \cdot 2\sqrt{2 - y^2}\, dy.$$

   The factor of 2 in front of the square root is because the horizontal slices extend from the left side of the circle to the right side, not just from the y-axis to the right side.

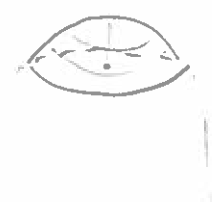
(e) Evaluate the integral in part (d). This should agree with your answer to part (c). Hint: Substitute $u = 2 - y^2$.

Solution: Following the hint, we let $u = 2 - y^2$, so $du = -2y\,dx$.

$$\begin{aligned}
\int_{y=1}^{y=\sqrt{2}} 4\pi y\sqrt{2-y^2}\,dy &= \int_{u=1}^{u=0} -2\pi\sqrt{u}\,du \\
&= \int_0^1 2\pi u^{1/2}\,du \\
&= 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^1 \\
&= \frac{4}{3}\pi.
\end{aligned}$$

(f) Roughly sketch the solid obtained by revolving the region around the y-axis.

Solution: It's the top of a sphere



(g) Set up the integral to find the volume of the solid from part (f) solid using cylindrical shells. Hint: The top half of the circle can be described as $y = \sqrt{2 - x^2}$.

Solution:

$$\int_0^1 2\pi x\left(\sqrt{2-x^2} - 1\right)\,dx$$

(h) Evaluate the integral from part (g). Hints: Substitute $u = 2 - x^2$. Recall that $2^{3/2} = 2^1 \cdot 2^{1/2} = 2\sqrt{2}$.

Solution: Following the hint, we let $u = 2 - x^2$, so $du = -2x\,dx$.

$$\begin{aligned}
\int_{x=0}^{x=1} 2\pi x\left(\sqrt{2-x^2}-1\right)\,dx &= \int_{u=2}^{u=1} -\pi\left(\sqrt{u}-1\right)\,du \\
&= \int_{1}^{2} \pi\left(u^{1/2}-1\right)\,du \\
&= \pi\left[\frac{2}{3}u^{3/2}-u\right]_1^2 \\
&= \pi\left(\frac{2}{3}\cdot 2\sqrt{2}-2-\frac{2}{3}+1\right) \\
&= \frac{4\sqrt{2}}{3}\pi-\frac{5}{3}\pi.
\end{aligned}$$

(i) Set up the integral to find the volume of the solid from part (f) using disks. Hint: The right half of the circle can be described as $x = \sqrt{2-y^2}$.
Solution:

$$\int_1^{\sqrt{2}} \pi\left(\sqrt{2-y^2}\right)^2\,dy$$

(j) Evaluate the integral in part (i). This should agree with your answer to part (h). Hint: Recall that $(\sqrt{2})^3 = \sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2} = 2\sqrt{2}$.
Solution:

$$\begin{aligned}
\int_1^{\sqrt{2}} \pi\left(2-y^2\right)\,dy &= \pi\left[2y-\frac{1}{3}y^3\right]_1^{\sqrt{2}} \\
&= \pi\left(2\sqrt{2}-\frac{1}{3}\cdot 2\sqrt{2}-2+\frac{1}{3}\right) \\
&= \frac{4\sqrt{2}}{3}\pi-\frac{5}{3}\pi.
\end{aligned}$$

(k) Extra credit: Explain (geometrically) why your answer to part (c) or (e), plus twice your answer to part (h) or (j), plus 2π, should equal $\frac{4}{3}\pi(\sqrt{2})^3 = \frac{8\sqrt{2}}{3}\pi$. Check that your answers satisfy this.
Solution: We can decompose the sphere of radius $\sqrt{2}$ into two "caps," a "napkin ring," and a cylinder of radius 1 and height 2, hence of volume 2π. And indeed we have

$$\frac{4}{3}\pi+\frac{8\sqrt{2}}{3}\pi-\frac{10}{3}\pi+2\pi=\frac{8\sqrt{2}}{3}\pi.$$