Solutions to Final Exam

Do problems 1 and 2, and **any two** of problems 3 to 6. Each problem is worth 20 points total. You will not get extra credit for doing more than two differential equations problems.

1. (a) Evaluate the *indefinite* integral

$$\int \frac{1}{(2x-1)^2}\,dx.$$

Solution: Let $u = 2x - 1$, so $\frac{du}{dx} = 2$, so $\frac{1}{2}du = dx$. Then the integral becomes

$$\begin{aligned}\int \tfrac{1}{2}u^{-2}\,du &= -\tfrac{1}{2}u^{-1} + C \\ &= -\tfrac{1}{2}(2x-1)^{-1} + C \text{ or } -\frac{1}{2}\cdot\frac{1}{2x-1} + C.\end{aligned}$$

If you skip the u-substitution but get the $-1/2$ right, that's fine. And I won't be uptight about the $+C$ in this problem.

(b) Take the derivative of your answer to part (a), using the chain rule. Is it $\frac{1}{(2x-1)^2}$? If not, revise your answer to part (a).

Solution: The derivative is

$$-\tfrac{1}{2}\cdot -(2x-1)^{-2}\cdot 2 = (2x-1)^{-2}.$$

(c) Evalutate

$$\int_1^\infty \frac{1}{(2x-1)^2}\,dx.$$

Solution: We get

$$\left[-\tfrac{1}{2}(2x-1)^{-1}\right]_1^\infty = (-\tfrac{1}{2}\cdot 0) - (-\tfrac{1}{2}\cdot 1) = \tfrac{1}{2},$$

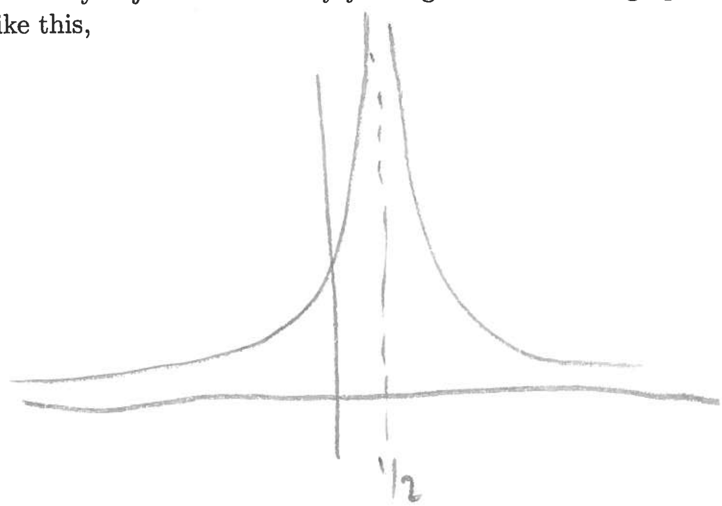
where for the first term we needed to observe that

$$\lim_{x\to\infty}\frac{1}{2x-1} = 0.$$

(d) Extra credit: Sketch the graph of $y = \frac{1}{(2x-1)^2}$. Then evalutate

$$\int_0^1 \frac{1}{(2x-1)^2}\, dx.$$

Solution: If you just do it blindly you'll get -1. But the graph looks like this,



so the answer had better not be negative. Instead, we observe that the function blows up at $x = \frac{1}{2}$, so we do the integral it in two pieces:

$$\int_0^{1/2} \frac{1}{(2x-1)^2}\, dx = \left[-\tfrac{1}{2}(2x-1)^{-1}\right]_0^{1/2} = \infty$$

because

$$\lim_{x \to \frac{1}{2}^-} \frac{1}{2x-1} = -\infty,$$

and

$$\int_{1/2}^1 \frac{1}{(2x-1)^2}\, dx = \left[-\tfrac{1}{2}(2x-1)^{-1}\right]_{1/2}^1 = \infty$$

because

$$\lim_{x \to \frac{1}{2}^+} \frac{1}{2x-1} = \infty.$$

So the whole integral from 0 to 1 is ∞.

2. On the board you see a sketch of the graph $y = x^{3/2}$.

(a) Set up an integral (but do not evaluate it!) to find the length of the part of the curve that lies between the points $(0,0)$ and $(1,1)$.

Solution: We have $y' = \frac{3}{2}x^{1/2}$, so

$$1 + (y')^2 = 1 + \tfrac{9}{4}x,$$

so the integral is

$$\int_{x=0}^{x=1} \sqrt{1 + \tfrac{9}{4}x}\; dx.$$

(b) Evaluate the integral you set up in part (a).

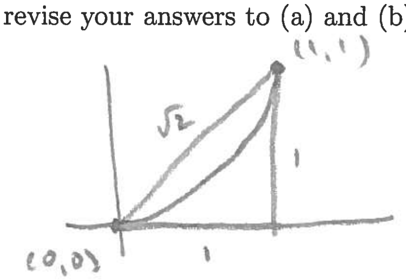
Hint: It should be an easy u-substitution, not anything crazy. At the end it's probably better to get a decimal approximation than to spend time simplifying.

Solution: Let $u = 1 + \frac{9}{4}x$, so $\frac{du}{dx} = \frac{9}{4}$, so $\frac{4}{9}du = dx$. If $x = 0$ then $u = 1$. If $x = 1$ then $u = 1 + \frac{9}{4} = \frac{13}{4}$. Thus the integral becomes

$$\begin{aligned}\int_{u=1}^{u=13/4} u^{1/2} \cdot \tfrac{4}{9}\, du &= \left[\tfrac{4}{9} \cdot \tfrac{2}{3} \cdot u^{3/2}\right]_1^{13/4} \\ &= \tfrac{8}{27}\left(\left(\tfrac{13}{4}\right)^{3/2} - 1\right) \\ &= \tfrac{8}{27}\left(\tfrac{13\sqrt{13}}{8} - 1\right) \\ &= \tfrac{1}{27}(13\sqrt{13} - 8) \\ &\approx 1.4397.\end{aligned}$$

(c) Extra credit: Draw a picture to explain why your answer should be between $\sqrt{2} \approx 1.4142$ and 2. Check that your answer lies in this range. If not, revise your answers to (a) and (b).

Solution:



We see that the curve is longer than the diagonal, whose length is $\sqrt{2}$, but shorter than the horizontal line plus the vertical line, whose combined length is 2.

3. You iron a cotton shirt using a iron heated to 400°F. You switch off the iron and leave it to cool in a room whose temperature is 70°F. After three minutes, the iron has cooled to 200°F.

   (a) Let y denote the temperature of the iron in degrees Fahrenheit, and let t denote the time in minutes. Write an equation involving dy/dt that expresses Newton's law of cooling: the rate at which the iron cools is proportional to the difference between the termperature of the iron and the termperature of the room.

   Solution:
$$\frac{dy}{dt} = k(y - 70)$$
   for some constant k.

   (b) Separate variables and integrate to get an equation involving y, t, and some constants. You can solve for y if you want, but you don't have to.

   Solution: Separating variables, we get
$$\frac{dy}{y - 70} = k\,dt.$$
   Integrating, we get
$$\ln|y - 70| = kt + C$$
   for some constant C. Observe that $y - 70$ will be positive in our situation, so we can omit the absolute value.

   (c) Plug in the data to determine the constants in part (b).

   Solution: When $t = 0$, we have $y = 400$, so
$$C = \ln(330).$$
   When $t = 3$, we have $y = 200$, so
$$\ln(130) = 3k + \ln(330),$$
   so
$$k = \frac{\ln(130) - \ln(330)}{3} \approx -.31.$$

(d) What will be the temperature of the iron one minute later – that is, four minutes after you switched it off?

Solution: When $t = 4$, we have

$$\ln(y - 70) \approx -.31 \cdot 4 + \ln(330) \approx 4.56,$$

so

$$y - 70 \approx e^{4.56} \approx 96,$$

so

$$y \approx 166°\text{F}.$$

(e) You decide to put the iron away in the closet when it cools down to 100°F. At what time will that happen?

Solution: When $y = 100$, we have

$$\ln 30 \approx -.31 \cdot t + \ln(330),$$

so

$$t = \frac{\ln(30) - \ln(330)}{-.31} \approx 7.7$$

minutes after we switched off the iron.

4. Based on §7.5 #10.

The department of fish and wildlife stocked a lake with 400 fish and estimated the carrying capacity to be 10,000. The number of fish tripled in the first year.

(a) Let y denote the number of fish in the lake, and let t denote the time in years. Assume a logistic growth model: that is, the rate at which the population grows is proportional to both the number of fish *and* how far the number is below the carrying capacity. Write an equation involving dy/dt to express this.

Solution: I would write

$$\frac{dy}{dt} = ky(10000 - y).$$

The book would write

$$\frac{dy}{dt} = ky(1 - y/10000).$$

Either one is acceptable – they describe the same behavior, you'll just get different k's.

(b) Separate variables and integrate to get an equation involving y, t, and some constants. You can solve for y if you want, but you don't have to.

Solution: Separating variables, we get

$$\frac{dy}{y(10000 - y)} = k\,dt.$$

Using partial fractions (details omitted), we rewrite the left-hand side as

$$\left(\frac{1/10000}{y} + \frac{1/10000}{10000 - y}\right) dt.$$

Integrating, we get

$$\tfrac{1}{10000}\ln|y| - \tfrac{1}{10000}\ln|10000 - y| = kt + C.$$

It is convenient to multiply through by 10000 to get

$$\ln|y| - \ln|10000 - y| = 10000kt + 10000C,$$

and then to rename $10000k$ as k' or ℓ, and $10000C$ as C' or D, to get

$$\ln|y| - \ln|10000 - y| = \ell t + D.$$

But you could make other choices.

(c) Plug in the data to determine the constants in part (b).

Solution: When $t = 0$, we have $y = 400$, so

$$D = \ln(400) - \ln(9600) \approx -3.18.$$

When $t = 1$, we have $y = 1200$, so

$$\ln(1200) - \ln(8800) = \ell - 3.18,$$

so

$$\ell \approx 1.19.$$

(d) How long will it take for the population to increase to 5,000?

Solution: When $y = 5000$, we have

$$\ln(5000) - \ln(5000) \approx 1.19t - 3.18.$$

The left-hand side is zero, so we find that

$$t \approx \frac{3.18}{1.19} \approx 2.7$$

years after the fish were first released.

(e) Due to a miscommunication with another agency, more fish were released into the lake, bringing the population to an unsustainable 15,000. How long will it take for the population to fall to 12,000?

Hint: Use the same differential equation and the same constant of proportionality k that you found earlier, but find a new C to reflect the new situation. Be careful with signs and absolute values.

Solution: We retain the equation

$$\ln|y| - \ln|10000 - y| = \ell t + D$$

and the constant

$$\ell \approx 1.19.$$

Now, at a new $t = 0$, we have $y = 15000$, so

$$\ln(15000) - \ln(5000) = 1.19 \cdot 0 + D,$$

so

$$D \approx 1.10.$$

Now if $y = 12000$ we have

$$\ln(12000) - \ln(2000) \approx 1.19t + 1.10,$$

so

$$t \approx \frac{\ln(12000) - \ln(2000) - 1.10}{1.19} \approx .58 \text{ years}$$

after the second release of fish.

5. Based on §7.3 #44 or last Friday's quiz.

A certain small country has \$10 billion in paper currency in circulation. Each day, \$50 million comes into the country's banks, and the same amount goes back out into circulation. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks.

(a) Let x denote the amount of new currency in circulation, in millions of dollars, and let t denote the time in days after the new currency started being introduced. Write an equation involving dx/dt to model the situation described above.

Hint: This is like a "salt in a tank" problem.

Solution: As I explained by e-mail on Monday,

$$\frac{dx}{dt} = 50 - 50 \cdot \frac{x}{10000} = 50 - \frac{x}{200}.$$

Of the currency that's in circulation, the fraction that's new is x million/10 billion $= x/10000$. Every day, 50 (million) times that fraction is taken out of circulation, and 50 (million) in new currency is put into circulation, which gives the equation above.

(b) Separate variables and integrate to get an equation involving x, t, and a constant. You can solve for x if you want, but you don't have to.

Solution: Separating variables, we get

$$\frac{dx}{50 - x/200} = dt.$$

Integrating, we get

$$-200 \ln |50 - x/200| = t + C$$

for some constant C.

(c) Plug in the data to determine the constant in part (b).

Solution: When $t = 0$, we have $x = 0$, so

$$C = -200 \ln(50) \approx -782.$$

(d) How much new currency is in circulation after 30 days?
Solution: When $t = 30$, we have

$$-200 \ln |50 - x/200| = 30 - 782,$$

so

$$\ln |50 - x/200| \approx 3.76,$$

so

$$50 - x/200 \approx e^{3.76} \approx 43.0,$$

so

$$x \approx 1400 \text{ million} = 1.4 \text{ billion}.$$

Our naive guess, if no new currency had circulated back to the bank, would have been $30 \cdot 50 = 1500$ million, so this is reasonable.

(e) How long will it take for the new bills to account for 90% of the currency in circulation?
Solution: If $x = 9000$ then

$$-200 \ln |50 - 9000/200| \approx t - 782,$$

so

$$t \approx -200 \ln(5) + 782 \approx 460 \text{ days},$$

so about a year and a quarter.

6. According to Toricelli's law, the rate at which water drains from a bath tub is proportional to the *square root* of the depth of the water. A rectangular bath tub was initially filled to a depth of 12 inches, and took 6 minutes to drain. (You can make up a width and length for the tub but it doesn't affect the answer.)

(a) Let y denote the depth of the water, and let t denote the time in minutes. Write an equation involving dy/dt to model the situation described above.
Solution:

$$\frac{dy}{dt} = k\sqrt{y}$$

for some constant k.

(b) Separate variables and integrate to get an equation involving y, t, and some constants. You can solve for y if you want, but you don't have to.

Solution: We separate variables to get

$$y^{-1/2}\,dy = k\,dt.$$

We integrate to get

$$2y^{1/2} = kt + C$$

for some constant C.

(c) Plug in the data to determine the constant in part (b).

Solution: When $t = 0$ we have $y = 12$, so

$$C = 2\sqrt{12} = \approx 6.9.$$

When $t = 6$ we have $y = 0$, so

$$0 = 6k + 6.9,$$

so

$$k = \frac{-6.9}{6} \approx -1.2.$$

(d) At what time was the bath tub half full?

Sanity check: it should be earlier than 3 minutes, because the first half drains faster than the second half.

Solution: When $y = 6$ we have

$$2\sqrt{6} \approx -1.2t + 6.9,$$

so

$$t \approx \frac{2\sqrt{6} - 6.9}{-1.2} \approx 1.7 \text{ minutes},$$

which agrees with our sanity check.