Solutions to Midterm 2

1. (a) Split up
$$\frac{3}{2x^2+x-1} = \frac{3}{(2x-1)(x+1)}$$
into partial fractions, as in §5.7.

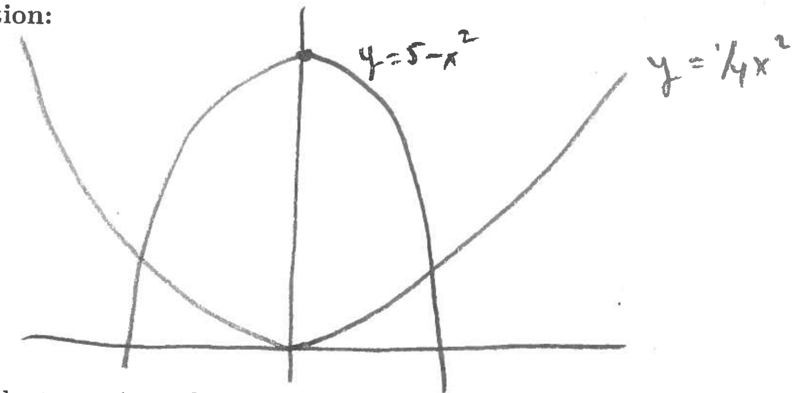
Solution: We seek numbers A and B such that
$$\frac{3}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}.$$
Clearing denominators, we get
$$3 = A(x+1) + B(2x-1).$$
Setting $x = -1$, we get $3 = -3B$, so $B = -1$.
Setting $x = \frac{1}{2}$, we get $3 = \frac{3}{2}A$, so $A = 2$. Thus
$$\frac{3}{2x^2+x-1} = \frac{2}{2x-1} - \frac{1}{x+1}.$$

(b) Evaulate
$$\int_1^2 \frac{3}{2x^2+x-1}\,dx.$$

Solution:
$$\begin{aligned}\int_1^2 \left(\frac{2}{2x-1} - \frac{1}{x+1}\right) dx &= \Big[\ln|2x-1| - \ln|x+1|\Big]_1^2 \\ &= (\ln 3 - \ln 3) - (\ln 1 - \ln 2) \\ &= \ln 2.\end{aligned}$$

2. (a) Sketch the parabolas $y = \frac{1}{4}x^2$ and $y = 5 - x^2$.

Solution:



(b) Find the two points where they meet

Solution: We set

$$\tfrac{1}{4}x^2 = 5 - x^2,$$

so

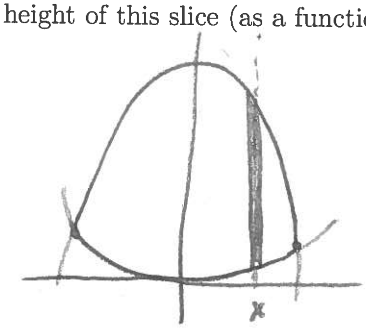
$$\begin{aligned} x^2 &= 20 - 4x^2 \\ 5x^2 &= 20 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

Thus the two points are

$$(\pm 2, 1).$$

(c) Sketch a thin vertical slice of the region enclosed by the two parabolas. Find the height of this slice (as a function of x).

Solution:



The height of the slice is $(5 - x^2) - (\frac{1}{4}x^2) = 5 - \frac{5}{4}x^2$.

(d) Find the area of the region by evaluating an integral that involves your answer to part (c).

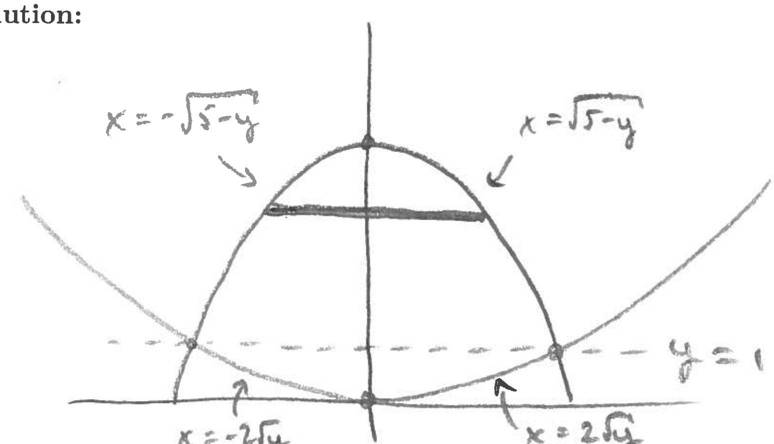
Solution: The region is symmetric left-to-right, so we can find the area of the right half and double it:

$$
\begin{aligned}
2\int_{x=0}^{x=2} \left(5 - \tfrac{5}{4}x^2\right)\,dx &= 2\left[5x - \tfrac{5}{12}x^3\right]_0^2 \\
&= 2\left(10 - \tfrac{10}{3} - 0\right) \\
&= \tfrac{40}{3} \text{ or } 13 + \tfrac{1}{3}.
\end{aligned}
$$

(e) Sketch a thin horizontal slice of the region. Find the width of this slice (as a function of y).

Hint: You will have one answer for the lower part of the region, and a different answer for the upper part.

Solution:



If $y \leq 1$ then the sides of the region are given by $y = \frac{1}{4}x^2$, so $x^2 = 4y$, so the right side is $x = 2\sqrt{y}$ and the left side is $x = -2\sqrt{y}$. Thus the width of the slice is

$$4\sqrt{y}.$$

If $y \geq 1$ then the sides of the region are given by $y = 5 - x^2$, so $x^2 = 5 - y$, so the right side is $x = \sqrt{5 - y}$ and the left side is $x = -\sqrt{5 - y}$. Thus the width of the slice is

$$2\sqrt{5 - y}.$$

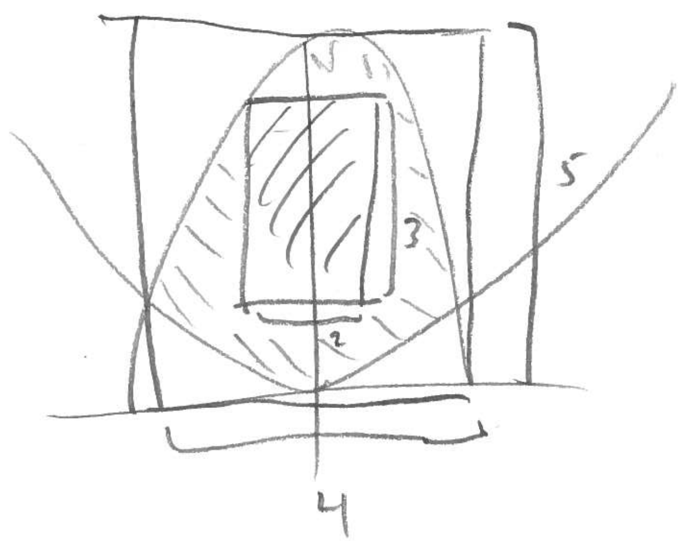
(f) Find the area of the region by evaluating an integral that involves your answer to part (e). (Really it's two integrals, because your answer to part (e) had two cases.)

Solution:

$$\begin{aligned}
\int_{y=0}^{y=1} 4y^{1/2}\,dy + \int_{y=1}^{y=5} 2(5-y)^{1/2}\,dy \\
&= 4 \cdot \tfrac{2}{3} y^{3/2}\Big|_0^1 + 2 \cdot \tfrac{-2}{3}(5-y)^{3/2}\Big|_1^5 \\
&= \tfrac{8}{3}(1-0) - \tfrac{4}{3}(0 - 4^{3/2}) \\
&= \tfrac{8}{3} + \tfrac{4}{3} \cdot 8 \\
&= \tfrac{40}{3} \text{ or } 13 + \tfrac{1}{3}.
\end{aligned}$$

(g) What sanity checks can you do to see if your answers to parts (d) and (f) are plausible?

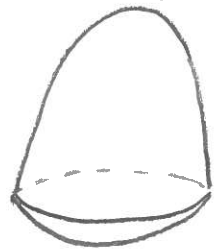
Solution: The two answers should agree, and be positive. The region is contained in a 5×4 rectangle, so the answer should be less than 20. And I can see a 3×2 rectangle that fits inside, so the answer should be more than 6.



3. (a) Sketch the solid obtained by revolving the region from problem 2 around the y-axis.

Optional: Compare it to a vegetable, toy, or other familiar object.

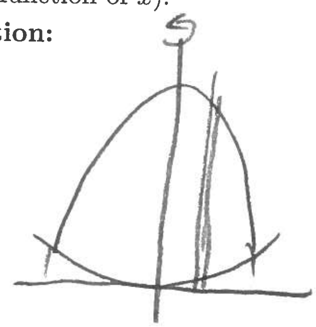
Solution:

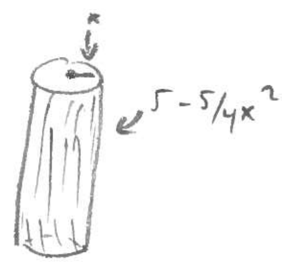


To me it looks like a plastic Easter egg that splits open.

(b) If you take the vertical slice from 2(c) and revolve it around the y-axis, it makes a thin cylidrical shell. Find the area of the side (as a function of x).

Solution:





The radius of the cylinder is x, so the circumference is $2\pi x$. The height is $5 - \frac{5}{4}x^2$, as we found earlier. Thus the area of the side is

$$2\pi x(5 - \tfrac{5}{4}x^2).$$

(c) Set up, *but do not evaluate*, an integral to find the volume of the solid using cylindrical shells.

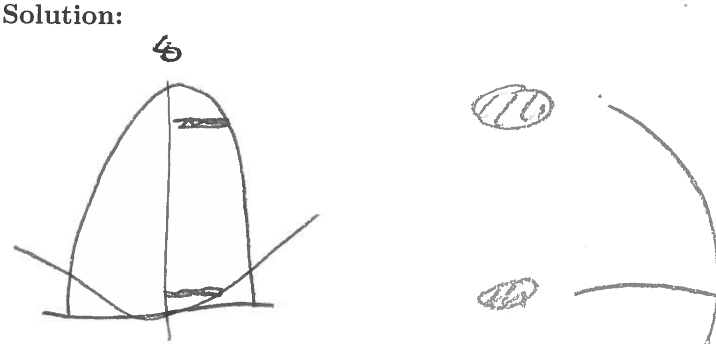
Solution:

$$\int_{x=0}^{x=2} 2\pi x(5 - \tfrac{5}{4}x^2)\,dx.$$

(d) If you take the horizontal slice from 2(e) and revolve it around the y-axis, it makes a thin disc. Find the area of the disc (as a function of y, again treating the lower and upper parts separately).

Hint: Be careful about radius versus diameter.

Solution:



If $y \geq 1$ then the radius of the disc is $\sqrt{5-y}$ – that is, half of your answer to 3(e) – so the area is

$$\pi(\sqrt{5-y})^2 = \pi(5-y).$$

If $y \leq 1$ then the radius is $2\sqrt{y}$, so the area is

$$\pi(2\sqrt{y})^2 = 4\pi y.$$

(e) Set up, *but do not evaluate*, an integral to find the volume of the solid using discs. (Again it's really a sum of two integrals.)

Solution:

$$\int_{y=0}^{y=1} 4\pi y\, dy + \int_{y=1}^{y=5} \pi(5-y)\, dy.$$