1. (10 points) Does \( \sum_{n=1}^{\infty} \frac{2n!}{n!^2} \) converge or diverge, and why?

Applying the ratio test, we have

\[
\lim_{n \to \infty} \frac{|2n+2|! |n+1|!}{|2n|! |n|!^2} = \lim_{n \to \infty} \frac{(2n+2)!}{(2n)!} \cdot \left(\frac{n!}{|n+1|!}\right)^2
\]

\[
= \lim_{n \to \infty} \frac{(2n+1)(2n+2)n!}{n!(n+1)n!} \cdot \frac{n!}{(n+1)n!}
\]

\[
= \lim_{n \to \infty} \frac{2n+1}{n+1} \cdot \frac{n!}{n^2}
\]

\[
= \lim_{n \to \infty} \frac{4n^2 + 6n + 2}{n^2 + 2n + 1}
\]

Now we either use L'Hôpital's rule twice or divide the top and bottom of the fraction by \( n^2 \), and either way we find that the limit is 4. Since this is greater than 1, the series diverges.
2. (10 points) For what values of x does the series \( \sum_{n=1}^{\infty} \frac{x^n}{3n^2} \) converge?

First we take absolute values and apply the ratio test, which gives

\[
\lim_{n \to \infty} \frac{|x|^{n+1}/3|n+1|^2}{|x|^n/3n^2} = \lim_{n \to \infty} \frac{|x|^{n+1} \cdot 3n^2}{|x|^n \cdot 3|n+1|^2} = \lim_{n \to \infty} \frac{|x| \cdot 3n^2}{3n^2 + 6n + 3}
\]

We see that the part involving n goes to 1, so the whole limit goes to \(|x|\). Thus the series converges if \(|x|<1\), diverges if \(|x|>1\), and if \(|x|=1\) then we need to do more work.

If \(x=1\) then we’re talking about \( \sum_{n=1}^{\infty} \frac{1}{3n^2} \), which converges by the integral test (since \(2>1\)).

If \(x=-1\) then we’re talking about \( \sum_{n=1}^{\infty} \frac{(-1)^n}{3n^2} \), which converges absolutely, because if we throw away the signs then we get the thing that we just said converges.

Thus the series converges for \(-1 \leq x \leq 1\).
Here is the form of Taylor’s theorem that we proved and have been using. Fix some \( x > 0 \), and suppose we find some \( M \) such that \( |f^{(d+1)}(t)| \leq M \) for all \( t \) between 0 and \( x \). Then the difference between \( f(x) \) and the \( d \)th Taylor polynomial

\[
f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(d)}(0)}{d!}x^d
\]

is at most \( \frac{Mx^{d+1}}{d!} \).

3. On the last practice midterm, you computed several derivatives of \( f(x) = e^{-x} \):

\[
f'(x) = -e^{-x} \quad f''(x) = e^{-x} \quad f'''(x) = -e^{-x} \quad f^{(4)}(x) = e^{-x} \quad f^{(5)}(x) = -e^{-x}
\]

From this you found that the Taylor series was \( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots \).

a) (5 points) Use a calculator to evaluate the fifth Taylor polynomial at \( x = 1 \).

\[
1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = 0.36666\ldots
\]

b) (5 points) For every \( d \) we have \( |f^{(d+1)}(t)| = e^{-t} \), and we see that if \( t \geq 0 \) then \( e^{-t} \leq 1 \), so in Taylor’s theorem we can take \( M = 1 \). How far, at most, does the theorem say that the number you found in part (a) can be from the true value of \( f(1) \)?

If \( x = 1 \), \( d = 5 \), and \( M = 1 \), then

\[
\frac{Mx^{d+1}}{d!} = \frac{1}{5!} = \frac{1}{120} = 0.0083333\ldots
\]

c) (5 points) Take your answer to part (a) plus your answer to part (b), and then your answer to part (a) minus your answer to part (b), to get upper and lower estimates for \( f(1) \).

Between 0.3583333\ldots and 0.375.

d) (5 points) Use a calculator to get a more exact value for \( f(1) = e^{-1} = \frac{1}{e} \).

If this isn’t in the range that you found in part (c), go back and fix any mistakes.

I get 0.367879441171442\ldots, which is in the right range.

e) If you want Taylor’s theorem to guarantee an error less than \( 10^{-4} = \frac{1}{10,000} \), what value of \( d \) should you take?

We want \( \frac{1}{d!} \leq \frac{1}{10,000} \), so \( d! \geq 10,000 \).

We find that \( 7! = 5,040 \) but \( 8! = 40,320 \), so we take \( d = 8 \).
4. The point of this problem is to approximate \( \int_0^1 \frac{\ln(x+1)}{x} \, dx \), which cannot be found by the methods of math 252.

a) (5 points) We have seen that the Taylor series for \( \ln(x+1) \) is \( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \).

Manipulate this to get the Taylor series for \( \frac{\ln(x+1)}{x} \).

Dividing by \( x \), we get

\[
1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots
\]

b) (5 points) Use your answer to part (a) to find \( \int_0^1 \frac{\ln(x+1)}{x} \, dx \).

(Your answer will be a series of numbers, not a power series.)

\[
\left. \left( x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} - \cdots \right) \right|_0^1 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots
\]

c) (5 points) Use a calculator to get an approximate value for the series in part (b). The true value is \( \frac{\pi^2}{12} = 0.822467033424113 \ldots \); if your answer is far from this, go back and fix any mistakes.

Going out to the \( \frac{1}{25} \) term, I get 0.83861111\ldots which is within 2% of the true answer.
5. This problem asks you to solve the differential equation \( y'' = y \) using power series.

a) (5 points) Suppose that \( y = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6 + \cdots \).

Find \( y' \) and \( y'' \).

\[
y' = c_1 + 2 c_2 t + 3 c_3 t^2 + 4 c_4 t^3 + 5 c_5 t^4 + 6 c_6 t^5 + \cdots
\]

\[
y'' = 2 c_2 + 3 \cdot 2 c_3 t + 4 \cdot 3 c_4 t^2 + 5 \cdot 4 c_5 t^3 + 6 \cdot 5 c_6 t^4 + \cdots
\]

b) (5 points) By equating the constant terms of \( y'' \) and \( y \), then the coefficients of \( t \), then the coefficients of \( t^2 \) and so on, solve for \( c_2, c_3, \) and so on up to \( c_6 \) in terms of \( c_0 \) and \( c_1 \).

From the constant terms we get \( 2c_2 = c_0 \), so \( c_2 = \frac{c_0}{2} \).

From the coefficients of \( t \) we get \( 3 \cdot 2 c_3 = c_1 \), so \( c_3 = \frac{c_1}{3 \cdot 2} \).

From the coefficients of \( t^2 \) we get \( 4 \cdot 3 c_4 = c_2 \), so \( c_4 = \frac{c_2}{4 \cdot 3} = \frac{c_0}{4 \cdot 3 \cdot 2} \).

From the coefficients of \( t^3 \) we get \( 5 \cdot 4 c_5 = c_3 \), so \( c_5 = \frac{c_3}{5 \cdot 4} = \frac{c_0}{5 \cdot 4 \cdot 3 \cdot 2} \).

From the coefficients of \( t^4 \) we get \( 6 \cdot 5 c_6 = c_4 \), so \( c_6 = \frac{c_4}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \).

c) (5 points) Write out the sixth Taylor polynomial of the particular solution that satisfies the initial conditions \( y(0) = 1 \) and \( y'(0) = -1 \).

(The point is that these initial conditions determine \( c_0 \) and \( c_1 \), which determine the rest.)

We have \( y(0) = c_0 \) and \( y'(0) = c_1 \), so these initial conditions give \( c_0 = 1 \) and \( c_1 = -1 \). Thus the sixth Taylor polynomial is

\[
y = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!}.
\]

d) Extra credit (5 points): Do you recognize your answer to part (c) as the Taylor series of a familiar function? Can you verify that it satisfies \( y'' = y \)? What if we had taken \( y(0) = 1 \) and \( y'(0) = 1 \)? What about \( y(0) = 1 \) and \( y'(0) = 0 \)?

Looking back at problem 3, we see that \( y = e^{-t} \), so \( y' = -e^{-t} \), so \( y'' = e^{-t} = y \) as desired.

If \( y(0) = 1 \) and \( y'(0) = 1 \) then we find that \( y = e^t \), which also satisfies \( y'' = y \).

To solve \( y(0) = 1 \) and \( y'(0) = 0 \) we can take the average of these:

\[
y = \frac{e^t + e^{-t}}{2} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots
\]

This is known as the hyperbolic cosine and is denoted \( \cosh(t) \).