1. The function $\text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt$, called the sine integral, has applications to signal processing.

   a) Try to evaluate this integral in the usual way, by making a $u$-substitution, integrating by parts, or whatever you can think of. Explain what you tried and where you got stuck. If you think you’ve succeeded, double-check by taking the derivative of your answer.

   b) We have seen that the Taylor series for $\sin(t)$ is

   $$t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

   Divide by $t$ and integrate from 0 to $x$ to get the Taylor series for $\text{Si}(x)$.

   c) Use this to get an approximate value for $\text{Si}(1)$. The true value is $0.946083070367183\ldots$

2. Approximating $\pi$. In lecture we took the geometric series $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$, substituted $r = -x^2$ to get

   $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

   and integrated to get the Taylor series for arctangent:

   $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

   Then we noticed that $\tan \frac{\pi}{4} = 1$, so we plugged in $x=1$ and multiplied by 4 to get an approximation to $\pi$. But the series converged very slowly: after 100 terms we only had $\pi$ to one decimal place.

   We also noticed that $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and wondered if this would yield a better approximation to $\pi$. Use a calculator to plug $x = \frac{1}{\sqrt{3}}$ into the Taylor series for arctangent above, carry it out to 4 terms and then 5 terms, and multiply by 6. How good an approximation do you get to $\pi = 3.14159265358989323846\ldots$?

(Continued on the next page.)
3. This problem asks you to solve the differential equation $y' = -2ty$ using power series.

a) Suppose that $y = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6 + \cdots$
   Find $y'$ and $-2ty$.

b) By equating the constant terms, then the coefficients of $t$, then the coefficients of $t^2$ and so on, solve for $c_1$, $c_2$, $c_3$ and so on in terms of $c_0$.

c) Take what you found in part (b) and write out the power series for $y$ that solves the differential equation.

d) By know you know the Taylor series for $e^x$:
   
   $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

   Substitute $x = -t^2$ and notice that your answer to part (c) is $c_0 e^{-t^2}$. Check that this satisfies $y' = -2ty$.

4. “Bessel functions” appear often in applied mathematics and physics, especially in analyzing the vibration of bells, cymbals, and drums. The Bessel function of order 0 is a solution to the differential equation $ty'' + y' + ty = 0$.

a) Suppose that $y = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6 + \cdots$
   Find $ty$, $y'$, and $ty''$.

b) Add them up to get $ty'' + y' + ty$, grouping together the constant terms, the terms involving $t$, the terms involving $t^2$, and so on; stop at $t^5$.

c) Setting your answer to part (b) equal to zero determines the coefficients $c_1$, $c_2$, $c_3$ and so on in terms of $c_0$. Set the constant term equal to zero and solve for $c_1$. Set the the coefficient of $t$ equal to zero and solve for $c_2$ in terms of $c_0$. Keep going up to $t^5$, which determines $c_6$.

d) Suppose that $y(t)$ satisfies the differential equation $ty'' + y' + ty = 0$ and the initial condition $y(0) = 1$ (which determines $c_0$). Write out the sixth Taylor polynomial of $y(t)$. Give an approximation to $y(1)$. 