1. Find a formula for the general term $a_n$ in the following sequences. Indicate whether you’re starting from $n=1$ or $n=0$; either choice is ok.

   a) $2, 5, 8, 11, 14, \ldots$

      If you start from $n=1$, you’ll get $a_n = 3n-1$.

      If you start from $n=0$, you’ll get $a_n = 3n+2$.

   b) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \ldots$

      There are many equivalent ways to write the answer.

      If you start from $n=1$, you’ll get $a_n = \frac{(-1)^{n-1}}{2^n}$ or something equivalent.

      If you start from $n=0$, you’ll get $a_n = \frac{(-1)^n}{2^{n+1}}$ or something equivalent.

2. Suppose that $a_1 = 2$, and for $n \geq 2$ we have $a_n = 3a_{n-1}$.

   a) Write out the first five terms of the sequence.

      2, 6, 18, 54, 162.

   b) Find an explicit formula for $a_n$.

      $a_n = 2 \cdot 3^{n-1}$
3. Evaluate the following limits:

a) \[ \lim_{n \to \infty} \frac{n^2 + 2n + 3}{3n^2 + 4n + 5} \]

We see that the limit is of the form \( \frac{\infty}{\infty} \).

One possibility is to multiply the top and bottom by \( \frac{1}{n^2} \), which gives

\[
\lim_{n \to \infty} \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{\frac{3}{n^2} + \frac{4}{n} + \frac{5}{n^2}} = \frac{1 + 0 + 0}{3 + 0 + 0} = \frac{1}{3}.
\]

The other is to use L’Hôpital’s rule twice:

\[
\lim_{n \to \infty} \frac{n^2 + 2n + 3}{3n^2 + 4n + 5} = \lim_{n \to \infty} \frac{2n + 2}{6n + 4} = \lim_{n \to \infty} \frac{2}{6} = \frac{1}{3}.
\]

b) \[ \lim_{n \to \infty} \frac{n}{(\ln n)^2} \]

Again we see that the limit is of the form \( \frac{\infty}{\infty} \).

Applying L’Hôpital’s rule once, we get \( \lim_{n \to \infty} \frac{1}{2\ln n} \cdot \frac{1}{n} \).

Simplifying, this becomes \( \lim_{n \to \infty} \frac{n}{2\ln n} \), which is again of the form \( \frac{\infty}{\infty} \).

Applying L’Hôpital’s rule again, we get \( \lim_{n \to \infty} \frac{1}{2/n} = \lim_{n \to \infty} n = \infty. \)
4. Consider the series \( \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots \).

a) Write it in sigma notation, that is, as \( \sum_{n=1}^{\infty} (\text{something}) \) or \( \sum_{n=0}^{\infty} (\text{something}) \).

Reusing the answer to problem 1a, we get

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}.
\]

b) Find the first three partial sums \( S_1, S_2, S_3 \).

\[
S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \quad S_3 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = \frac{3}{8}.
\]

c) Does the series converge or diverge? If it converges, find the sum.

Hint: It is a geometric series, although it doesn’t start from 1.

We know that if \( |r| < 1 \) then \( 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \).

Thus \( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \frac{1}{1+1/2} = \frac{1}{3/2} = \frac{2}{3} \).

Thus the series we’re considering can either be found as \( \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \), or as \( 1 - \frac{2}{3} = \frac{1}{3} \).
5. Consider the telescoping series \( \sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1}). \)

a) Find the first three partial sums \( S_1, S_2, S_3. \)

\[
S_1 = \sqrt{1} - \sqrt{0} = 1
\]
\[
S_2 = (\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) = \sqrt{2}
\]
\[
S_3 = (\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) = \sqrt{3}
\]

b) Give a formula for the \( n^{\text{th}} \) partial sum \( S_n. \)

\[ S_n = \sqrt{n} \]

c) Does the series converge or diverge? If it converges, find the sum.

The limit of the partial sums is \( \lim_{n \to \infty} \sqrt{n} = \infty, \) so the series diverges.

6. Use the integral test to decide whether the following series converge or diverge.

a) \( \sum_{n=1}^{\infty} \frac{1}{(2n+5)^2} \)

To evaluate \( \int_{1}^{\infty} \frac{1}{(2x+5)^2} \, dx, \) which we saw on the very first quiz, we substitute \( u = 2x + 5, \)

so \( du = 2 \, dx, \) so \( dx = \frac{1}{2} \, du, \) so the integral becomes

\[
\int_{1}^{\infty} \frac{1}{2u^2} \, du = \int_{1}^{\infty} \frac{1}{2u^2} \, du = \frac{1}{2} \cdot \frac{u^{-1}}{-1} \bigg|_{1}^{\infty} = 0 - \left(-\frac{1}{14}\right) = \frac{1}{14}.
\]

This is finite, so the sum converges.

b) \( \sum_{n=1}^{\infty} \frac{|\ln n|^2}{n} \)

To evaluate \( \int_{1}^{\infty} \frac{|\ln x|^2}{x} \, dx, \) we substitute \( u = \ln x, \) so \( du = \frac{1}{x} \, dx, \) so the integral becomes

\[
\int_{0}^{\infty} u^2 \, du = \frac{u^3}{3} \bigg|_{0}^{\infty} = \infty - 0 = \infty.
\]

Thus the sum diverges.